Math 501 HW 3 Shiying.

#0: If $\alpha : \mathbb{R} \to \mathbb{R}^3$ is on a sphere of radius R, we have

$$|\alpha(s)| = R \qquad \forall s$$

$$\Rightarrow \quad d_s |\alpha(s)|^2 = 0 = 2\langle \alpha(s), \alpha'(s) \rangle$$

Taking another derivative we get

$$\langle \alpha'(s), \alpha'(s) \rangle + \langle \alpha(s), \alpha''(s) \rangle = 0$$

which means

$$\langle \alpha(s), \alpha''(s) \rangle = -1 .$$

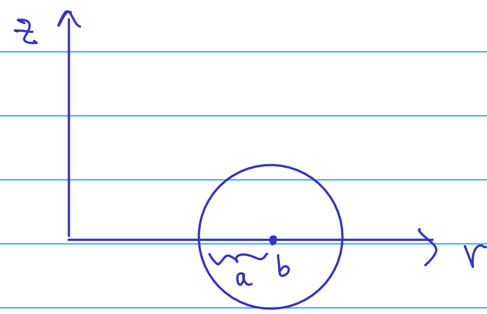
Since $\left|\langle \alpha(s), \alpha''(s) \rangle\right| \le |\alpha(s)| \cdot |\alpha''(s)|$

$$K(s) = |\alpha''(s)| \geq 1/|\alpha(s)| = 1/R \qquad \square$$

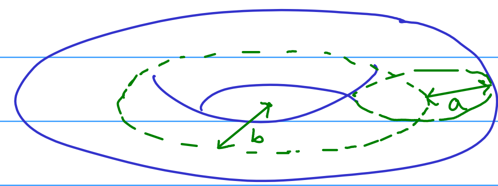
#1. Let $r = a\cos\theta + b$. Since $0 < a < b$, $r > 0$.

We have $\begin{cases} x^2 + y^2 = r^2 \\ (r-b)^2 + z^2 = a^2 \end{cases}$ $\leftarrow$ cylindrical coordinate

On r-z "half plane" we have a circle centered at $(b, 0)$ with radius a.



The original surface is simply an evolution of this i.e., a torus.



Three of the following problems (# 2, 3, 5) show the following are equivalent (all functions are smooth)

1) S is regular surface
2) S is locally graph of $z = f(y,z)$ or $y = g(x,z)$ or $x = h(y,z)$.
3). S is locally $f^{-1}(a)$ for some $f(x,y,z)$ and regular value a.

Here 2) ⇒ 3) is trivial.

problem 2 : 2) ⇒ 1)

3 : 3) ⇒ 1)

6 : 1) ⇒ 2)

#2 Let the graph be S. $S = S \cap V$ for any open $V \supseteq S$.

Define $g : U \to S$ by $g(x,y) = (x, y, f(x,y))$

g is smooth since f is.

Define $\pi : S \to U$ by $\pi(x,y,z) = (x,y)$

π is projection hence continuous.

We have $\pi \circ g = id$ on U, $g \circ \pi = id$ on S

⇒ g is homeomorphism

Furthermore

$$dg = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ \partial_x f & \partial_y f \end{pmatrix} \begin{pmatrix} dx \\ dy \end{pmatrix}$$

Since the matrix has rank 2, dg is injective.

Therefore S is a regular surface.

#3. a is a regular value of f ⇒ $df \neq 0$ at $\forall p \in f^{-1}(a)$

⇒ one of $\partial_x f(p)$, $\partial_y f(p)$, $\partial_z f(p)$ is $\neq 0$

Say $\partial_z f(p) \neq 0$. We can use Implicit Function theorem and get $z = g(x,y)$ for some smooth g locally.

Using problem 2 we are done.

If we don't use Implicit Function Theorem, we can define a smooth map

$g: U \to \mathbb{R}^3$ as $g(x,y,z) = (x, y, f(x,y,z))$

Since $dg = \begin{pmatrix} 1 & 0 & \partial_x f \\ 0 & 1 & \partial_y f \\ 0 & 0 & \partial_z f \end{pmatrix} \begin{pmatrix} dx \\ dy \\ dz \end{pmatrix}$ and $\partial_z f(p) \neq 0$

we have dg is invertible at p. Using Inverse Function Thm

$\exists$ open nbh of p U_p and open $W_p \subseteq \mathbb{R}^3$ s.t.

$g|_{U_p}$ is a diffeomorphism between U_p & W_p.

Let $V_p \subseteq \mathbb{R}^2$ be $V_p = W_p \cap \{z = a\}$

then $g^{-1}(V_p) = f^{-1}(a) \cap U_p$

Since g^{-1} is diffeomorphism, so is $g^{-1}|_{V_p}$. Hence $f^{-1}(a)$ is regular surface.

#4. $f(x,y,z) = x^2/a^2 + y^2/b^2 + z^2/c^2$

$df = 2x/a^2 \, dx + 2y/b^2 \, dy + 2z/c^2 \, dz$

For it to be non surjective, we have to have

$x = y = z = 0$, which means $f(x,y,z) = 0$

Therefore 0 is the only critical value of f.

Using the result of #3, $f^{-1}(1)$ is a regular surface

#5. S is regular surface & $p \in S$ $\Rightarrow$ $\exists$ open $U \subseteq \mathbb{R}^2$ & $p \in V \subseteq \mathbb{R}^3$ s.t $W: U \to V \cap S$ is smooth homeo with injective dW_q.

Since

$$dW = \begin{pmatrix} \partial_u X & \partial_v X \\ \partial_u y & \partial_v y \\ \partial_u z & \partial_v z \end{pmatrix} \begin{pmatrix} du \\ dv \end{pmatrix}$$

For it to be one to one one of the following must have rank 2 : $\begin{pmatrix} \partial_u x & \partial_v x \\ \partial_u y & \partial_v y \end{pmatrix}_p$, $\begin{pmatrix} \partial_u x & \partial_v x \\ \partial_u z & \partial_v z \end{pmatrix}_p$, $\begin{pmatrix} \partial_u y & \partial_v y \\ \partial_u z & \partial_v z \end{pmatrix}_p$

Say it's $\begin{pmatrix} \partial_u x & \partial_v x \\ \partial_u y & \partial_v y \end{pmatrix}_p$ i.e., $\left|\begin{pmatrix} \partial_u x & \partial_v x \\ \partial_u y & \partial_v y \end{pmatrix}_p\right| \neq 0$.

Let $W' = \pi \circ W$, where $\pi : \mathbb{R}^3 \to \mathbb{R}^2$ has $\pi(x, y, z) = (x, y)$

$dW'(\pi(p)) = \begin{pmatrix} \partial_u x & \partial_v x \\ \partial_u y & \partial_v y \end{pmatrix}_p$ is invertible.

We can use Inverse Function Thm and get an open nbh $V_{\pi(p)} \subseteq \mathbb{R}^2$ of $\pi(p)$, an open set $U_p \subseteq U$ s.t that $W'\big|_{U_p}$ is a diffeomorphism between $V_{\pi(p)}$ & U_p. Let $V_p = V_{\pi(p)} \times \mathbb{R} \cap S$

It's again open in S. In V_p, we have

$$z = z(u, v) \quad \text{with} \quad (u, v) \in U_p.$$

Let $(W')^{-1}$ on $V_{\pi(p)}$ denoted by $(x, y) \mapsto (u(x, y), v(x, y))$ we have $z = z\big(u(x, y), v(x, y)\big)$ $\forall z \in V_p$.

All the functions involved are smooth.