Math 501 HW # 4 Shiying

14 A tangent vector along $u=$ Constant is X_v &

" " " $v=$ constant is X_u.

Therefore

$$\theta(u,v) = \frac{\langle X_u, X_v\rangle}{|X_u|\cdot|X_v|} = \frac{F}{\sqrt{EG}}.$$

15. $|X_u \times X_v|$ measures the area of the parallelogram bounded by X_u & X_v. Therefore if well defined, the quantity

$$\int_{U_0} |X_u \times X_v| \, du \, dv$$

can serve as a definition of area of $V_0 = X(U_0)$.

The well defined-ness simply means independence on parametrization. Say we have another parametrization $\tilde{X}: \tilde{U}_0 \to S$ s.t $V_0 = \tilde{X}(\tilde{U}_0)$. Then we have

$$du\,dv = \left|\frac{\partial(u,v)}{\partial(\tilde{u},\tilde{v})}\right| d\tilde{u}\,d\tilde{v}$$

and $$|\tilde{X}_{\tilde{u}} \times \tilde{X}_{\tilde{v}}| = \left|(\partial_{\tilde{u}} u\, X_u + \partial_{\tilde{u}} v\, X_v) \times (\partial_{\tilde{v}} u\, X_u + \partial_{\tilde{v}} v\, X_v)\right|$$

$$= |\partial_{\tilde{u}} u\, \partial_{\tilde{v}} v - \partial_{\tilde{u}} v\, \partial_{\tilde{v}} u|\, |X_u \times X_v|$$

$$= \left|\frac{\partial(u,v)}{\partial(\tilde{u},\tilde{v})}\right| |X_u \times X_v|$$

$$\Rightarrow |\tilde{X}_{\tilde{u}} \times \tilde{X}_{\tilde{v}}|\, d\tilde{u}\,d\tilde{v} = |X_u \times X_v|\, du\,dv$$

Hence this definition of area is justified.

We also recognize that

$$|X_u \times X_v| = |X_u|\,|X_v|\sqrt{1-\cos^2\theta(u,v)}$$

$$= \left[|X_u|^2 |X_v|^2 - \langle X_u, X_v\rangle^2\right]^{1/2} = \sqrt{EG-F^2}$$

For S^2, we use (θ, φ) coordinate in $\begin{cases} \varepsilon \le \theta \le \pi - \varepsilon \\ \varepsilon \le \varphi \le 2\pi - \varepsilon \end{cases}$

Then $A(S^2) = \lim_{\varepsilon \to 0} \int_{\varepsilon}^{2\pi-\varepsilon} \int_{\varepsilon}^{\pi-\varepsilon} \sqrt{EG - F^2}\, d\theta\, d\varphi$

$$= \lim_{\varepsilon \to 0} \int_{\varepsilon}^{2\pi-\varepsilon} \int_{\varepsilon}^{\pi-\varepsilon} \sin\theta\, d\theta\, d\varphi = 4\pi.$$

Comment: In real life we usually don't bother to write down the $\lim_{\varepsilon \to 0}$ process. It's OK with manifolds that are smooth. But we should keep in mind that the surface formula works on a closed subset inside a coordinate patch.

#16. See do Carmo P. 99.

#1 (do Carmo) Straightforward. Skipped.

#5 (do Carmo) The surface R can be parametrized as

$$X : Q \to R \quad \text{by} \quad (x, y) \mapsto (x, y, z(x, y))$$

$X_x = (1, 0, f_x)$, $X_y = (0, 1, f_y)$

$X_x \times X_y = (-f_x, -f_y, 1)$

$$\Rightarrow \quad A = \iint_Q |X_x \times X_y|\, dx\, dy = \iint_Q \sqrt{1 + f_x^2 + f_y^2}\, dx\, dy.$$

#14. (a) By definition we have

$$\langle \operatorname{grad} f, a X_u + b X_v \rangle = a f_u + b f_v, \quad \forall a, b$$

Let $\operatorname{grad} f = c X_u + d X_v$. Then

$$(c, d) \begin{pmatrix} E & F \\ F & G \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = (f_u, f_v) \begin{pmatrix} a \\ b \end{pmatrix}, \quad \forall \begin{pmatrix} a \\ b \end{pmatrix}$$

Hence $(c, d) = (f_u, f_v) \begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1}$

$$= (f_u, f_v) \frac{1}{EG - F^2} \begin{pmatrix} G & -F \\ -F & E \end{pmatrix}$$

$$= \frac{1}{EG - F^2} \left(f_u G - f_v F, \; -f_u F + f_v E \right)$$

In particular when $S = \mathbb{R}^2$, $E = G = 1$, $F = 0$
$(c, d) = (f_u, f_v)$

(b). $df(v) = \langle \text{grad} f, v \rangle = |\text{grad} f| \, |v| \cos\theta$
where θ is the angle between grad$f$ & v.
Hence $df(v)_{max} = |\text{grad} f|$ achived when $v = \dfrac{\text{grad} f}{|\text{grad} f|}$.

(c) $\text{grad} f \neq 0$ means $(f_u, f_v) \neq (0, 0)$
$\Rightarrow$ any pt in $\mathbb{R}$ is a regular value of f
Analogous to what we did in the previous HW, $f^{-1}(c)$
is regular for any c.

For any $v \in T_pC \subseteq T_pS$, $\langle \text{grad} f, v \rangle = df(v) = 0$
by definition of C. $\Rightarrow$ grad$f$ is normal to C.

15. (a). $\varphi(u, v) = \text{const} \Rightarrow \varphi'(u, v) = u' \varphi_u + v' \varphi_v = 0$
for any parametrization t of the curve.
Since φ is regular, $(\varphi_u, \varphi_v) \neq (0, 0)$.
Therefore a nonzero $V \in T_p C_\varphi$ can be chosen as
$\varphi_v X_u - \varphi_u X_v$.
Similarly a nonzero $V \in T_p C_\psi$ can be chosen as
$\psi_v X_u - \psi_u X_u$

φ, ψ curve normal to each other iff

$\langle \varphi_v X_u - \varphi_u X_v , \psi_v X_u - \psi_u X_u \rangle = 0$

which means $E\varphi_v \psi_v - F(\varphi_v \psi_u + \varphi_u \psi_v) + G\varphi_u \psi_u = 0$

(b) $\varphi = v\cos u$, $\varphi_u = -v\sin u$, $\varphi_v = \cos u$

$\psi = (v^2+a^2)\sin^2 u$, $\psi_u = 2(v^2+a^2)\sin u\cos u$, $\psi_v = 2v\sin^2 u$.

$E = v^2+a^2$. $F = 0$. $G = 1$

$E\varphi_v \psi_v - F(\varphi_v \psi_u + \varphi_u \psi_v) + G\varphi_u \psi_u$

$= (v^2+a^2)\cdot 2v\sin^2 u\cos u - 2(v^2+a^2)\sin^2 u\cos u = 0$

$\Rightarrow$ φ & ψ constants curves are orthogonal.