Math 501 HW # 5 Shiying

3. It's easy to calculate, or see directly that
$\vec{N}(x) = \frac{1}{r}\vec{x}$, $\Rightarrow dN = \frac{1}{r} Id$.

5. (1) & (2) Straightforward calculation. skipped.

(3) At $u=v=0$, $dN_p(x_u) = (2,0,0)$, $dN_p(x_v) = (0,-2,0)$
$\Rightarrow dN_p\left(v_1 x_u + v_2 x_v\right) = 2v_1 x_u + 2v_2 x_v$
In matrix form it's simply
$$dN_p\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

6. Exactly as 5. Skipped.

7. (a) $\langle A(v), w\rangle = \langle v, A(w)\rangle \quad \forall\, v, w \in \mathbb{R}^n$
$\Leftrightarrow \langle A(e_i), e_j\rangle = \langle e_i, A(e_j)\rangle \quad \forall\, e_i, e_j$ orthonormal basis

$\Leftrightarrow A_{ij} = A_{ji} \quad \forall\, i, j$

(b) $(\Rightarrow)$ $B(v,w) = \langle A(v), w\rangle$
A linear, $\langle\, ,\rangle$ binear $\Rightarrow$ B bilinear
A self-adjoint $\Rightarrow$ $B(v,w) = \langle A(v), w\rangle = \langle v, A(w)\rangle$
$= \langle A(w), v\rangle = B(w,v)$.
$(\Leftarrow)$ Given a symmetric bilinear form B. define a linear map A as $A(v) = \sum_{i=1}^{n} B(v, e_i)\, e_i$
Since B is bilinear, A is linear. Moreover
$\langle A(v), w\rangle = \sum_{i=1}^{n} B(v, e_i)\,\langle e_i, w\rangle = \sum_{i=1}^{n} B(v, e_i)\cdot w_i$
$= \sum_{i,j} B(e_j, e_i)\, w_i v_j$

$= \sum_{i,j} B(e_i, e_j)\, w_i\, v_j$ B is symmetric

$= \langle A(W), V \rangle = \langle V, A(W) \rangle$.

(c) Use Lagrange multiplier. set

$F(V, \lambda) = \langle A(V), V \rangle - \lambda(\langle V, V \rangle - 1)$

We have

$\partial_V F(V, \lambda) = 2A(V) - 2\lambda V$

$\partial_\lambda F(V, \lambda) = 1 - \langle V, V \rangle$

For V that maximizes $\langle A(V), V \rangle$ with $\langle V, V \rangle = 1$

$2A(V) - 2\lambda V = 0$. $\Rightarrow$ $A(V) = \lambda V$.

(d). If $A(V) = \lambda V$ & $\langle V, W \rangle = 0$ then

$\langle V, A(W) \rangle = \langle A(V), W \rangle = \lambda \langle V, W \rangle = 0$

(e) We do by induction. The existence of e_1 s.t. $A(e_1) = \lambda e_1$ is ensured by (c) & the compactness of S^{n-1}. If have found orthonormal $\{e_1, \cdots, e_k\}$ s.t. $A(e_i) = \lambda_i e_i$ then in the complement of the subspace spanned by $\{e_1, \cdots e_k\}$, let it be V_{n-k}, for any $W \in V_{n-k}$ we have $\langle e_i, W \rangle = 0$. By (d) $\langle e_i, A(W) \rangle = 0$ $\forall i$. Hence $A(W) \in V_{n-k}$ as well. A restricts to a linear map of V_{n-k}. We can apply (c) and find a unit $e_{k+1} \in V_{n-k}$, s.t $A(e_{k+1}) = \lambda_{k+1} e_{k+1}$, Clearly $\{e_1, \cdots, e_{k+1}\}$ is an orthonormal eigen set. Proceed till we get $\{e_1, \cdots, e_n\}$.

8. Curvature of circle $= 1/\text{radius of circle}$

$k_G \cos\theta = k_S \Rightarrow K_S = K_G / \cos\theta$

9. $x \to X = (x, \frac{1}{2} a x^2, 0)$

$X' = (1, ax, 0)$, $X'' = (0, a, 0)$

$$K = \frac{|X' \times X''|}{|X'|^3} = \frac{a}{\sqrt{1+a^2x^2}^3}$$

at $x=0$ $K = a$.

11. See the proof on P. 145 of do Carmo.

12. Change of basis for matrix A is simply $A \to UAU^{-1}$ for some invertible U.

Since $tr(UAU^{-1}) = tr(U^{-1}UA) = tr A$ and $\det(UAU^{-1}) = \det(U)\det(A)\det(U)^{-1} = \det(A)$

tr & det are basis indept.

13. In a nice basis $dN_p = \begin{pmatrix} k_1 & 0 \\ 0 & k_2 \end{pmatrix}$

$K = k_1 k_2 = \det(dN_p)$, $H = -\frac{1}{2}(k_1+k_2) = -\frac{1}{2} tr(dN_p)$ under this basis. Since det & tr are basis indept, the equality is true in general.