Math 501 HW # 6 Shiying

15. As hinted, we first prove that if all normal lines of a plane curve pass through a pt, it's part of a circle. The pf is easy. Let that pt be p, p must be in the plane of the curve. WLOG let p be origin. We have $\vec{N}$ parallel to $\vec{x}$. since $\langle \vec{x}', N \rangle = 0$, $\langle \vec{x}', \vec{x} \rangle = 0$. Hence $(|\vec{x}|^2)' = 0$, $|\vec{x}|$ is a const.

Now go to the main problem. Pick $\forall\, p \in L$, and let P_p be the plane passing p perpendicular to L. Let $C_p = P_p \cap S$. If $C_p \neq \emptyset$ or $\{p\}$, pick any $q \in C_p$, $q \neq p$. Let l_q be the normal line of S passing q. By assumption l_q intersect L, say at r.
On the other hand, let $\vec{\tau}_q$ be the tangent direction of of C_p at q. We have $\vec{\tau}_q \perp l_q$ & $\vec{\tau}_q \perp L$ which leads to $\vec{\tau}_q \perp \vec{pq}$. Therefore the normal line of C_p at any q pass through p. Hence C_p is an arc of circle centered at p. Since p is arbitrary, S is a portion of revolution.

17. Use Taylor's thm

$$f(x) = f(0) + f'(0)x + \frac{1}{2} f''(0) x^2 + R(x)$$
$$= \frac{1}{2} f''(0) x^2 + R(x)$$

where $\lim_{x \to 0} |R(x)| / x^2 = 0$. Since $f''(0) > 0$, we can

pick $\frac{1}{4} f''(0)$, there $\exists\, \delta > 0$, s.t

$$\frac{|R(x)|}{x^2} < \frac{1}{4} |f''(0)|, \quad \forall \quad 0 < |x| < \delta$$

$$\Rightarrow f(x) > \frac{1}{2} f''(0) x^2 - \frac{1}{4} f''(0) x^2 = \frac{1}{4} f''(0) x^2 > 0$$

when $0 < |x| < \delta$.

19. $X(u,v) = (u, v, f(u,v))$

$X_u = (1, 0, \partial_u f)$, $\quad X_v = (0, 1, \partial_v f)$

$E = 1 + (\partial_u f)^2$, $F = \partial_u f\, \partial_v f$, $G = 1 + (\partial_v f)^2$

$EG - F^2 = A^2$, where $A = \sqrt{1 + (\partial_u f)^2 + (\partial_v f)^2}$

$N = (-\partial_u f, -\partial_v f, 1)/A$

$X_{uu} = (0, 0, \partial_u^2 f)$. $X_{uv} = (0, 0, \partial_u \partial_v f)$, $X_{vv} = (0, 0, \partial_v^2 f)$

$\Rightarrow e = \partial_u^2 f / A$. $f = \partial_u \partial_v f / A$, $g = \partial_v^2 f / A$

$K = (eg - f^2)/(EG - H^2) = \partial_u^2 f\, \partial_v^2 f - (\partial_u \partial_v f)^2 / A^4$

$H = \frac{1}{2}(eG - 2fF + gE)/(EG - F^2)$

$=$ Something not inspiring. See do Carmo p. 163

The rest is very tedious.

For sphere, $f = \sqrt{R^2 - u^2 - v^2}$, $\partial_u f = -\frac{u}{f}$, $\partial_v f = -\frac{v}{f}$.

$\partial_u^2 f = -\frac{1}{f} - \frac{u^2}{f^3}$, $\partial_u \partial_v f = -\frac{uv}{f^3}$. $\partial_v^2 f = -\frac{1}{f} - \frac{v^2}{f^3}$

$E = 1 + u^2/f^2 = \frac{R^2 - v^2}{R^2 - u^2 - v^2}$, $F = \frac{uv}{f^2} = \frac{uv}{R^2 - u^2 - v^2}$, $G = \frac{R^2 - u^2}{R^2 - u^2 - v^2}$

$A = R / \sqrt{R^2 - u^2 - v^2}$

$e = -\frac{1}{fA} E \qquad g = -\frac{1}{fA} G, \qquad f = -\frac{1}{fA} F$

$\quad = -\frac{1}{R} E \qquad\quad = -\frac{1}{R} G \qquad\quad = -\frac{1}{R} F$

The second funda form $= -\frac{1}{R}$ the first funda form

$\Rightarrow dN_p = \begin{pmatrix} \frac{1}{R} & 0 \\ 0 & \frac{1}{R} \end{pmatrix} \Rightarrow k_1 = k_2 = -\frac{1}{R}$, $K = \frac{1}{R^2}$, $H = -\frac{1}{R}$

20 do Carmo p. 167 Prop 2.