Math 501 HW #7 Shiying

5. Let $L(p) = \varphi(p) - \varphi(0)$

First of all $|L(p)| = |\varphi(p) - \varphi(0)| = |p|$

L is an isometry.

$\forall\, p, q$, $\quad |L(p) - L(q)| = |\varphi(p) - \varphi(q)| = |p-q|$

$$\Rightarrow |L(p+q) - L(p) - L(q)|^2 = L(p+q)^2 + L(p)^2 + L(q)^2 + 2L(p)\cdot L(q) - 2L(p+q)(L(p)+L(q))$$

$$= |L(p+q) - L(p)|^2 + |L(p+q) - L(q)|^2 - |L(p) - L(q)|^2$$

$$- L(p+q)^2 + L(p)^2 + L(q)^2$$

$$= (p+q-p)^2 + (p+q-q)^2 - (p-q)^2 + (p+q)^2 - p^2 - q^2 = 0$$

$\Rightarrow L(p+q) = L(p) + L(q) \quad \forall\, p, q.$

L is a linear isometry.

8. (a) From notes we get

$$A_1 = \Gamma^1_{11,v} + \Gamma^1_{11}\Gamma^1_{12} + \Gamma^2_{11}\Gamma^1_{22} + e a_{12}$$
$$- \left(\Gamma^1_{12,u} + \Gamma^1_{12}\Gamma^1_{11} + \Gamma^2_{12}\Gamma^1_{12} + f a_{11}\right)$$

Therefore

$A_1 = 0 \Rightarrow e a_{12} - f a_{11} = \Gamma^1_{12,u} - \Gamma^1_{11,v} + \Gamma^2_{12}\Gamma^1_{12} - \Gamma^2_{11}\Gamma^1_{22}$

Since $(a_{ij})^t = -\mathrm{II}\,\mathrm{I}^{-1}$ we have

$\mathrm{II}^{-1}(a_{ij})^t = -\mathrm{I}^{-1} \quad \Rightarrow$

$$\begin{pmatrix} g & -f \\ -f & e \end{pmatrix}\begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{11} \end{pmatrix} = -\frac{\det \mathrm{II}}{\det \mathrm{I}}\begin{pmatrix} G & -F \\ -F & E \end{pmatrix} = K\begin{pmatrix} -G & F \\ F & E \end{pmatrix}$$

Hence $-f a_{11} + e\, a_{12} = KF.$

(b). From notes we get

$C_1 = e_v + \Gamma^1_{11} f + \Gamma^2_{11} g - (f_u + \Gamma^1_{12} e + \Gamma^2_{12} f)$

The first M-C eqn follows from $C_1 = 0$.

(c) ~~&~~ (d) By symmetry, they are generated from $A_1 = B_1 = C_1 = 0$ with $1 \Leftrightarrow 2$, $u \Leftrightarrow v$, $E \Leftrightarrow G$, $e \Leftrightarrow g$.

(e) As I commented, let's study

$\langle N, X_u \rangle$, $\langle N, X_v \rangle$, $\langle N, N \rangle$

They're constants, 0, 0, 1 respectively. Therefore

$0 = \langle N, X_u \rangle_{uv} - \langle N, X_u \rangle_{vu} = \langle N_{uv} - N_{vu}, X_u \rangle + \langle N, X_{uuv} - X_{uvu} \rangle$

$0 = \langle N, X_v \rangle_{vu} - \langle N, X_v \rangle_{uv} = \langle N_{vu} - N_{uv}, X_v \rangle + \langle N, X_{vvu} - X_{vuv} \rangle$

and $0 = \langle N, N \rangle_{uv} - \langle N, N \rangle_{vu} = 2 \langle N_{uv} - N_{vu}, N \rangle = 2 C_3$

The third eqn says $C_3 = 0$ trivially

The first two say

$A_3 = B_3 = 0 \Leftrightarrow \langle N_{uv} - N_{vu}, X_u \rangle = \langle N_{uv} - N_{vu}, X_v \rangle = 0$

$\Leftrightarrow \langle N, X_{uuv} - X_{uvu} \rangle = \langle N, X_{vvu} - X_{uvv} \rangle = 0$

$\Leftrightarrow C_1 = C_3 = 0$. □

10. (a) $K = (eg - f^2)/(EG - F^2) = e/G = 1$.

(b) For C-symbol: $G_u = -2 \sin u \cos u$, Everything 0

$\Rightarrow \Gamma^1_{11} = \Gamma^2_{11} = \Gamma^1_{12} = \Gamma^2_{22} = 0$

$\Gamma^2_{12} = \frac{1}{2} G_u / G = -\tan u$, $\Gamma^2_{12,u} = -\frac{1}{\cos^2 u}$

$\Gamma^1_{22} = -\frac{1}{2} G_u = \sin u \cos u$

(c) In Gauss formula: $LHS = -1 = RHS = \Gamma^2_{12,u} + (\Gamma^2_{12})^2$ ✓

(d) In the first M-C eqn

LHS = 0 = RHS ✓

In the 2nd M-C eqn

LHS = 0 , RHS = $-\Gamma^2_{12}$ = $-\tan^2 u$. LHS ≠ RHS

Therefore such surface doesn't exist.