Math 501 - Differential Geometry
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January 12, 2012

1. CURVES

Definition. A map

$$F(x_1, ..., x_m) = (f_1(x_1,...,x_m) , ..., f_n(x_1,...,x_m))$$

from an open set in one Euclidean space into another Euclidean space is said to be ***smooth*** (or of class C^∞) if it has continuous partial derivatives of all orders.

In this chapter, we will be dealing with ***smooth curves***

$$\alpha : I \to R^3 ,$$

where $I = (a, b)$ is an open interval in the real line R^3, allowing $a = -\infty$ or $b = +\infty$.

Do Carmo calls these *"parametrized differentiable curves"*, to emphasize that the specific function α is part of the definition. Thus

$$\alpha(t) = (\cos t, \sin t) \quad \text{and} \quad \beta(t) = (\cos 2t, \sin 2t)$$

are considered to be different curves in the plane, even though their ***images*** are the same circle.

Examples.

(1) The helix $\alpha(t) = (a \cos t, a \sin t, bt)$, $t \in R$

(2) $\alpha(t) = (t^3, t^2)$.

Problem 1. Let $\alpha(t)$ be a smooth curve which does not pass through the origin. If $\alpha(t_0)$ is the point of its image which is closest to the origin (assuming such a point exists), and if $\alpha'(t_0) \neq 0$, show that the position vector $\alpha(t_0)$ is orthogonal to the velocity vector $\alpha'(t_0)$.

Problem 2. Let $\alpha: I \to R^3$ be a smooth curve and let $V \in R^3$ be a fixed vector. Assume that $\alpha'(t)$ is orthogonal to V for all $t \in I$ and also that $\alpha(t_0)$ is orthogonal to V for some $t_0 \in I$. Prove that $\alpha(t)$ is orthogonal to V for all $t \in I$.

Problem 3. Let $\alpha: I \to R^3$ be a smooth curve. Show that $|\alpha(t)|$ is constant if and only if $\alpha(t)$ is orthogonal to $\alpha'(t)$ for all $t \in I$.

Definition. A smooth curve $\alpha: I \to R^3$ is said to be ***regular*** if $\alpha'(t) \neq 0$ for all $t \in I$. Equivalently, we say that α is an ***immersion*** of I into R^3.

The curve $\alpha(t) = (t^3, t^2)$ in the plane fails to be regular when $t = 0$.

A regular smooth curve has a well-defined tangent line at each point, and the map α is one-to-one on a small neighborhood of each point $t \in I$.

Convention. For simplicity, we'll begin omitting the word *"smooth"*. So for example, we'll just say *"regular curve"*, but mean *"regular smooth curve"*.

Problem 4. If α: [a, b] $\rightarrow$ R^3 is just continuous, and we attempt to define the ***arc length*** of the image α[a, b] to be the LUB of the lengths of all inscribed polygonal paths, show that this LUB may be infinite.

By contrast, show that if α is of class C^1 (meaning that it has a first derivative $\alpha'(t)$ which is continuous), then this LUB is finite and equals $\int_a^b |\alpha'(t)| \, dt$.

Let $\alpha: I \rightarrow R^3$ be a regular (smooth) curve. Then the arc length along α, starting from some point $\alpha(t_0)$, is given by

$$s(t) = \int_{t_o}^{t} |\alpha'(t)|\, dt .$$

Note that $s'(t) = |\alpha'(t)| \neq 0$, so we can invert this function to obtain $t = t(s)$.

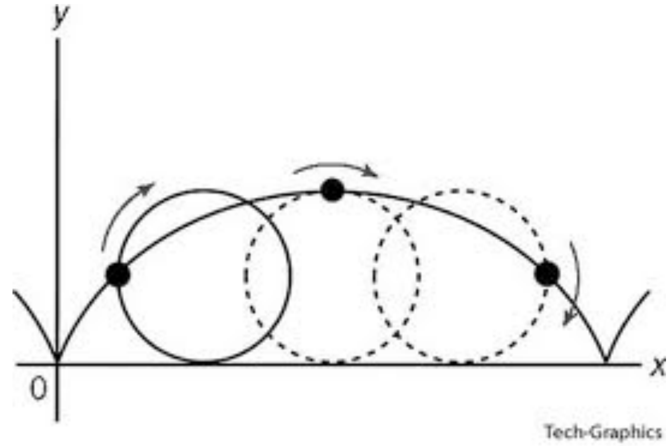
Then $\beta(s) = \alpha(t(s))$ is a reparametrization of our curve, and $|\beta'(s)| = 1$.

We will say that β is ***parametrized by arc length.***

In what follows, we will generally parametrize our regular curves by arc length.

If $\alpha: I \rightarrow R^3$ is parametrized by arc length, then the unit vector $T(s) = \alpha'(s)$ is called the ***unit tangent vector*** to the curve.

Problem 5. A circular disk of radius 1 in the xy-plane rolls without slipping along the x-axis. The figure described by a point of the circumference of the disk is called a ***cycloid.***



(a) Find a parametrized curve $\alpha: \mathbb{R} \to \mathbb{R}^2$ whose image is the cycloid.

(b) Find the arc length of the cycloid corresponding to a complete rotation of the disk.

Problem 6. Let $\alpha: [a, b] \rightarrow R^3$ be a parametrized curve, and set $\alpha(a) = p$ and $\alpha(b) = q$.

(1) Show that for any constant vector V with $|V| = 1$,

$$(q - p) \bullet V = \int_a^b \alpha'(t) \bullet V \, dt \leq \int_a^b |\alpha'(t)| \, dt .$$

(2) Set $V = (q - p) / |q - p|$ and conclude that

$$|\alpha(b) - \alpha(a)| \leq \int_a^b |\alpha'(t)| \, dt .$$

This shows that the curve of shortest length from $\alpha(a)$ to $\alpha(b)$ is the straight line segment joining these points.

Problem 7. Let $\alpha: I \to R^3$ be parametrized by arc length. Thus the tangent vector $\alpha'(s)$ has unit length. Show that the norm $|\alpha''(s)|$ of the second derivative measures the rate of change of the angle which neighboring tangents make with the tangent at s .

Definition. If $\alpha: I \to R^3$ is parametrized by arc length, then the number $\kappa(s) = |\alpha''(s)|$ is called the ***curvature*** of α at s .

Problem 8. Show that the curvature of a circle is the reciprocal of its radius.

Let $\alpha: I \to R^3$ be parametrized by arc length. When the curvature $\kappa(s) \neq 0$, the unit vector

$$N(s) = \alpha''(s) / |\alpha''(s)|$$

is well-defined.

Problem 9. Show that the unit vector $N(s)$ is normal to the curve, in the sense that $N(s) \bullet T(s) = 0$, where $T(s)$ is the unit tangent vector to the curve.

Definition. When $\kappa(s) \neq 0$, we call $N(s)$ the ***principal normal vector*** to the curve.

Let $\alpha: I \to R^3$ be parametrized by arc length, and let $T(s)$ be the unit tangent vector along α .

If the curvature $\kappa(s) \neq 0$, then we also have the principal normal vector $N(s)$ at $\alpha(s)$.

In that case, define the ***binormal vector*** $B(s)$ to α at s by the vector cross product,

$$B(s) = T(s) \times N(s) .$$

Problem 10. Show that $B'(s)$ is parallel to $N(s)$.

Definition. If $\kappa(s) \neq 0$, the ***torsion*** $\tau(s)$ of the curve α at s is defined by the formula

$$B'(s) = -\tau(s)\, N(s) .$$

This is the opposite sign convention from do Carmo.

Problem 11. Find the curvature and torsion of the helix

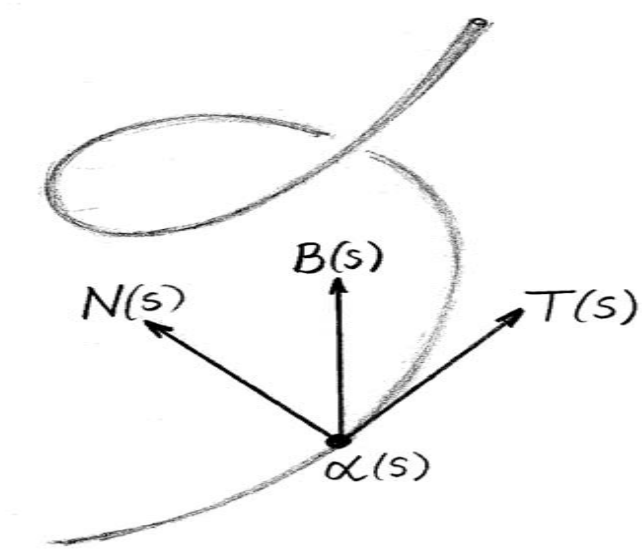
$$\alpha(t) = (a \cos t, a \sin t, b\, t) .$$

Problem 12. Let $\alpha: I \rightarrow R^3$ be parametrized by arc-length and have nowhere vanishing curvature $\kappa(s) \neq 0$. Show that

$$T'(s) = \kappa(s)\, N(s)$$

$$N'(s) = -\kappa(s)\, T(s) + \tau(s)\, B(s)$$

$$B'(s) = -\tau(s)\, N(s)\,.$$



Definition. The above equations are called the ***Frenet equations***, and the orthonormal frame

$$T(s)\ ,\ \ N(s)\ ,\ \ B(s)$$

is called the ***Frenet frame*** along the curve α .

THEOREM. ***Given smooth functions*** $\kappa(s) > 0$ ***and*** $\tau(s)$, ***for*** $s \in I$, ***there exists a regular curve*** $\alpha : I \to R^3$ ***parametrized by arc length, with curvature*** $\kappa(s)$ ***and torsion*** $\tau(s)$.

Moreover, another other such curve $\beta : I \to R^3$ ***differs from*** α ***by a rigid motion of*** R^3.

This result is sometimes called the

fundamental theorem of the local theory of curves.

Problem 13. The curvature of a smooth curve in the plane can be given a well-defined sign, just like the torsion of a curve in 3-space. Explain why this is so.

Problem 14. Given a smooth function $\kappa(s)$ defined for s in the interval I, show that the arc-length parametrized plane curve having $\kappa(s)$ as curvature is given by

$$\alpha(s) = \left(\int \cos\theta(s)\,ds + a,\ \int \sin\theta(s)\,ds + b\right),$$

where

$$\theta(s) = \int \kappa(s)\,ds + \theta_0 .$$

Show that this solution is unique up to translation by (a, b) and rotation by θ_0.

Proof of the fundamental theorem of the local theory of curves in $\mathbf{R}^3$.

We are given smooth functions $\kappa(s) > 0$ and $\tau(s)$, for $s \in I$, and must find a regular curve $\alpha: I \to R^3$ parametrized by arc length, with curvature $\kappa(s)$ and torsion $\tau(s)$.

Let's begin by writing the Frenet equations for the Frenet frame.

$$T'(s) = \kappa(s)\, N(s)$$

$$N'(s) = -\kappa(s)\, T(s) + \tau(s)\, B(s)$$

$$B'(s) = -\tau(s)\, N(s)\,.$$

We'll view this as a system of three first order linear ODEs, with given coefficients $\kappa(s)$ and $\tau(s)$, for the unknown Frenet frame $T(s)$, $N(s)$, $B(s)$.

We can also view it as a system of nine first order linear ODEs for the components of the Frenet frame.

Now the fundamental existence and uniqueness theorem for systems of first order ODEs promises a unique "local solution", that is, a solution defined in some unspecified neighborhood of any given point $s_0 \in I$, with preassigned "initial conditions" $T(s_0)$, $N(s_0)$, $B(s_0)$.

Although for general systems we can only guarantee a local solution, for linear systems another theorem promises a unique "global solution", that is, one defined on the entire interval I.

So we'll use that theorem, pick an arbitrary point $s_0 \in I$,
and pick an arbitrary "right handed" orthonormal frame
$T(s_0)$, $N(s_0)$, $B(s_0)$ to get us started.

Then we'll apply the global existence and uniqueness theorem for linear systems of ODEs to get a unique family of vectors $T(s)$, $N(s)$, $B(s)$ which are defined for all $s \in I$, which satisfy the Frenet equations, and which have arbitrary preassigned initial values $T(s_0)$, $N(s_0)$, $B(s_0)$.

Let's pause to check that the nature of the Frenet equations

$$T'(s) = \kappa(s)\, N(s)$$

$$N'(s) = -\kappa(s)\, T(s) + \tau(s)\, B(s)$$

$$B'(s) = -\tau(s)\, N(s)$$

guarantees that if we start off with an orthonormal frame $T(s_0)$, $N(s_0)$, $B(s_0)$, then the solution will be an orthonormal frame for all $s \in I$.

Consider the six real-valued functions defined for $s \in I$, and obtained by taking the various inner products of the vectors $T(s)$, $N(s)$, $B(s)$:

$\langle T(s), T(s)\rangle$	$\langle N(s), N(s)\rangle$	$\langle B(s), B(s)\rangle$
$\langle T(s), N(s)\rangle$	$\langle T(s), B(s)\rangle$	$\langle N(s), B(s)\rangle$.

When $s = s_0$, these six quantities start off with the values

1	1	1
0	0	0

These six quantities satisfy a system of first order linear ODEs, obtained from the Frenet equations. For example,

$$\langle T(s), T(s)\rangle ' = 2 \langle T(s), T'(s)\rangle = 2 \kappa(s) \langle T(s), N(s)\rangle$$

$$\langle T(s), N(s)\rangle ' = \langle T'(s), N(s)\rangle + \langle T(s), N'(s)\rangle$$

$$= \langle \kappa(s) N(s), N(s)\rangle + \langle T(s), -\kappa(s)T(s) + \tau(s) B(s)\rangle$$

$$= \kappa(s) \langle N(s), N(s)\rangle - \kappa(s) \langle T(s), T(s)\rangle + \tau(s) \langle T(s), B(s)\rangle ,$$

and so forth.

The constant solution

$$\langle T(s), T(s)\rangle = 1 \quad \langle N(s), N(s)\rangle = 1 \quad \langle B(s), B(s)\rangle = 1$$

$$\langle T(s), N(s)\rangle = 0 \quad \langle T(s), B(s)\rangle = 0 \quad \langle N(s), B(s)\rangle = 0$$

satisfies this system of ODEs, with the given initial conditions, so by uniqueness this is the only solution.

Conclusion: If the vectors $T(s)$, $N(s)$, $B(s)$ start out orthonormal at $s_0 \in I$, then they remain orthonormal for all $s \in I$.

Where are we so far?

We have proved that, given smooth functions $\kappa(s) > 0$ and $\tau(s)$ defined for all $s \in I$, and an orthonormal frame $T(s_0)$, $N(s_0)$, $B(s_0)$ defined for some $s_0 \in I$, then there is a unique orthonormal frame $T(s)$, $N(s)$, $B(s)$ defined for all $s \in I$ with these preassigned initial values, and satisfying the Frenet equations throughout I .

Now, to get the curve $\alpha: I \rightarrow R^3$ defined for $s \in I$ and having the preassigned curvature $\kappa(s) > 0$ and torsion $\tau(s)$, just pick the point $\alpha(s_0)$ at random in R^3 and then define

$$\alpha(s) = \alpha(s_0) + \int_{so}^{s} T(s)\, ds .$$

We get $\alpha'(s) = T(s)$, which is a unit vector, so α is parametrized by arc-length.

The Frenet equations

$$T'(s) = \kappa(s)\,N(s)$$

$$N'(s) = -\kappa(s)\,T(s) + \tau(s)\,B(s)$$

$$B'(s) = -\tau(s)\,N(s)$$

then tell us that the curve α has curvature $\kappa(s)$ and torsion $\tau(s)$, as desired.

Once the point $\alpha(s_0)$ and the initial orthonormal frame $T(s_0)$, $N(s_0)$, $B(s_0)$ is picked, the curve is unique.

Thus any other such curve $\beta: I \rightarrow R^3$ differs from α by a rigid motion of R^3 .

This completes the proof of the fundamental theorem of the local theory of curves in R^3 .

Problem 15. Let $\alpha: I \to R^3$ be a regular curve with nowhere vanishing curvature. Assume that all the principal normal lines of α pass through a fixed point in R^3. Prove that the image of α lies on a circle.

Problem 16. Let $r = r(\theta)$, $a \le \theta \le b$, describe a plane curve in polar coordinates.

(a) Show that the arc length of this curve is given by

$$\int_a^b [r^2 + (r')^2]^{1/2} \, d\theta .$$

(b) Show that the curvature is given by

$$\kappa(\theta) = [2(r')^2 - r\,r'' + r^2] / [(r')^2 + r^2]^{3/2} .$$

Problem 17. Let $\alpha: I \to R^3$ be a regular curve, not necessarily parametrized by arc length.

(a) Show that the curvature of α is given by

$$\kappa(t) = |\alpha' \times \alpha''| / |\alpha'|^3 .$$

(b) If the curvature is nonzero, so that the torsion is well-defined, show that the torsion is given by

$$\tau(t) = (\alpha' \times \alpha'') \bullet \alpha''' / |\alpha' \times \alpha''|^2 .$$

Definitions. A ***closed plane curve*** is a regular curve $\alpha: [a, b] \to R^2$ such that α and all its derivatives agree at a and at b, that is,

$$\alpha(a) = \alpha(b), \ \alpha'(a) = \alpha'(b), \ \alpha''(a) = \alpha''(b), \ldots .$$

Alternatively, one can use the entire real line as domain, $\alpha: R \to R^2$, and require that α be periodic of some period $L > 0$, that is, $\alpha(t + L) = \alpha(t)$ for all $t \in R$.

Another alternative: one can use a circle (of any radius) as the domain for a closed curve.

A curve is ***simple*** if it has no further intersections, other than the coincidence of the beginning and end points.

If we use a circle for the domain, $\alpha: S^1 \to R^2$, then the curve is simple if α is one-to-one. Since S^1 is compact, this is the same thing as saying that α is a homeomorphism onto its image.

If $\alpha: [a, b] \rightarrow R^2$ is a regular closed curve in the plane, parametrized by arc length, then its ***total curvature*** is defined by the integral

$$\text{Total curvature} = \int_a^b \kappa(s)\, ds .$$

Problem 18. (a) Show that the total curvature of a regular closed curve in the plane is $2n\pi$ for some integer n.

(b) Show that if the regular closed curve is simple, then $n = +1$ or -1.

(c) Suppose that a regular closed curve in the plane has curvature which is strictly positive or strictly negative, and that the above integer n equals $+1$ or -1.

Show that the curve is simple.

Let $\alpha: S^1 \rightarrow R^2$ be a regular closed curve in the plane. For each point $\theta \in S^1$, the unit tangent vector $T(\theta)$ to the curve at the point $\alpha(\theta)$ is given by

$$T(\theta) = \alpha'(\theta) / |\alpha'(\theta)| .$$

Thus $T: S^1 \rightarrow S^1$, and then the induced map

$$T_* : \pi_1(S^1) \rightarrow \pi_1(S^1)$$

is a group homorphism from the integers to the integers, and hence is multiplication by some integer n, which we call the ***degree*** of the map T, or the ***winding number*** or ***rotation index*** of the curve α.

Problem 19. Show that this integer n is the same as the integer n in the previous problem, that is, show that the total curvature of the curve α is $2\pi n$.

Definition. Let α_0 and $\alpha_1 : S^1 \to R^2$ be regular closed curves in the plane. A homotopy

$$A: S^1 \times [0, 1] \to R^2$$

between α_0 and α_1 is said to be a ***regular homotopy*** if each intermediate curve, $\alpha_t : S^1 \to R^2$, defined by $\alpha_t(\theta) = A(\theta , t)$, is a *regular* curve.

Remark. If α_0 and $\alpha_1 : S^1 \to R^2$ are regularly homotopic, then they have the same winding number.

WHITNEY-GRAUSTEIN THEOREM. Two regular curves α_0 and α_1: $S^1 \to R^2$ are regularly homotopic if and only if they have the same winding number.

Volumes of tubes...two problems.

(1) Show that the area of a tube of radius ε about a simple closed curve of length L in the plane is $2\varepsilon L$.

(2) Show that the volume of a tube of radius ε about a simple closed curve of length L in 3-space is $\pi \varepsilon^2 L$.

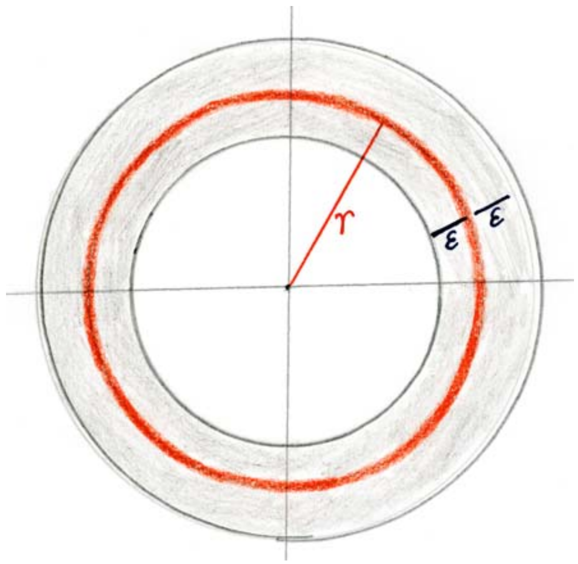
We will solve both of these problems, and the Frenet equations for curves will be our main tool.

Tubes about circles in the plane.

The simplest example is that of a tube of radius ε about a circle of radius r in the plane, so just an annulus between concentric circles of radii $r + \varepsilon$ and $r - \varepsilon$, with area

$$\pi (r + \varepsilon)^2 - \pi (r - \varepsilon)^2 = \pi\, 4r\varepsilon = (2\pi r)(2\varepsilon)$$

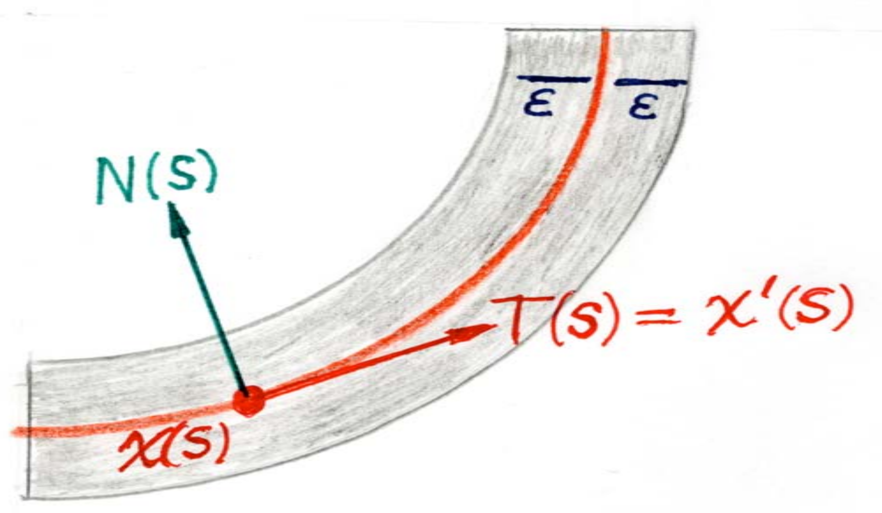
$$= \text{(circumference of circle) (width of tube)}$$



Tubes about any curve in the plane.

Parametrize the curve by arc length: $x = x(s)$ for $0 \leq s \leq L$.

Let $T(s) = x'(s)$ and $N(s)$ denote unit tangent and normal vectors along the curve.



Frenet eqns: $T'(s) = \kappa(s)\, N(s)$ and $N'(s) = -\kappa(s)\, T(s)$.

To produce the ε-tube about this curve, we define

$$F: \{0 \le s \le L\} \times \{-\varepsilon < t < \varepsilon\} \;\rightarrow\; R^2$$

by

$$F(s, t) \;=\; x(s) \;+\; t\, N(s)\ .$$

Then the partial derivatives of F are given by

$$F_s \;=\; x'(s) \;+\; t\, N'(s) \;=\; T(s) \;+\; t\,(-\kappa(s)\, T(s))$$

$$=\; (1 - t\,\kappa(s))\, T(s)$$

$$F_t \;=\; N(s)\ .$$

Hence the area of the ε-tube about our curve is given by

$$\int_s \int_t \ |\det dF| \ dt\, ds \ = \ \int_s \int_t (1 - t\, \kappa(s))\, dt\, ds$$

$$= \ \int_s \ (t - 1/2\, t^2\, \kappa(s))\Big|_{-\varepsilon}^{\varepsilon} \ ds \ = \ \int_s \ 2\varepsilon \ ds \ = \ L \cdot 2\varepsilon$$

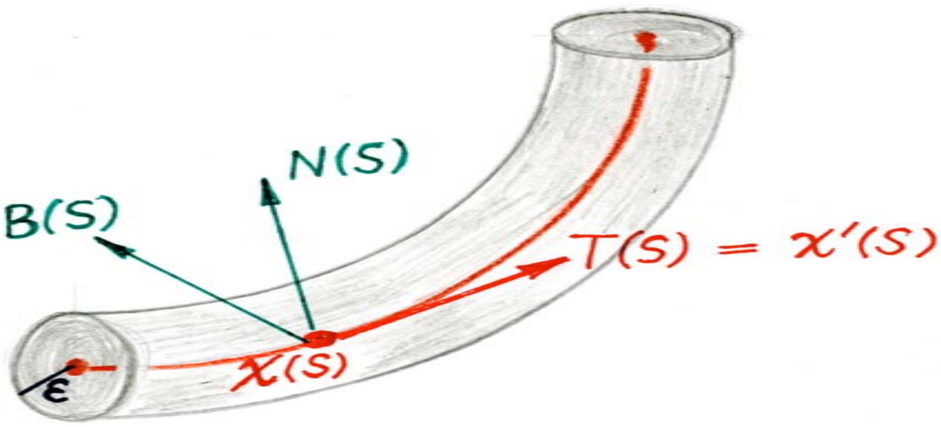
$$= \ (\text{length of curve})\ (\text{width of strip}),$$

independent of the nature of the curve.

Tubes about any curve in 3-space.

Parametrize the curve by arc length: $x = x(s)$ for $0 \leq s \leq L$.

Frenet frame along the curve: $T(s) = x'(s)$, $N(s)$, $B(s)$.



Frenet eqns:

$$\begin{aligned} T'(s) &= \kappa(s)\, N(s) \\ N'(s) &= -\kappa(s)\, T(s) + \tau(s)\, B(s) \\ B'(s) &= -\tau(s)\, N(s) \end{aligned}$$

To produce the ε-tube about this curve, we define

$$F: \{0 \le s \le L\} \times \{t^2 + u^2 < \varepsilon\} \;\rightarrow\; R^3$$

by

$$F(s, t, u) \;=\; x(s) \;+\; t\, N(s) \;+\; u\, B(s)\;.$$

Then the partial derivatives of F are given by

$$F_s \;=\; (1 - t\,\kappa(s)\; T(s) \;-\; u\,\tau(s)\, N(s) \;+\; t\,\tau(s)\, B(s)$$

$$F_t \;=\; N(s)$$

$$F_u \;=\; B(s)$$

Hence the volume of the ε-tube about our curve is given by

$$\int_s \int_{t^2+u^2<\varepsilon^2} |\det dF| \; dt\, du\, ds = \int_s \int_{t^2+u^2<\varepsilon^2} (1 - t\,\kappa(s))\, dt\, du\, ds$$

$$= \int_s \pi\,\varepsilon^2 \; ds = L \cdot \pi\,\varepsilon^2$$

$$= \text{(length of curve) (area of } \varepsilon\text{-disk)},$$

independent of the nature of the curve.

We used the fact that the integral of the odd function t over the disk $t^2 + u^2 < \varepsilon^2$ is zero.

Problem. To get a Frenet frame along a curve in R^3, one needs to assume that the curvature $\kappa(s)$ never vanishes.

Without this hypothesis, one can still prove that

$$\text{vol } \varepsilon\text{-tube} = (\text{length of curve})\,(\text{area of } \varepsilon\text{-disk})\,.$$

(a) Let $T(s)$, $A(s)$, $B(s)$ be an ON frame along our curve $x(s)$. Show that the Frenet eqns are replaced by

$$T'(s) = \alpha(s)\,A(s) + \beta(s)\,B(s)$$

$$A'(s) = -\alpha(s)\,T(s) + \gamma(s)\,B(s)$$

$$B'(s) = -\beta(s)\,T(s) - \gamma(s)\,A(s)\,.$$

(b) Defining the ε-tube about our curve $x(s)$ by

$$F(s, t, u) = x(s) + t\,A(s) + u\,B(s),$$

with $T(s) = x'(s)$, and hence $A(s)$ and $B(s)$ orthogonal to the curve, show that we get

$$\text{vol } \varepsilon\text{-tube} = \int_s \int_{t^2+u^2<\varepsilon^2} |\det dF|\; dt\, du\, ds$$

$$= \int_s \int_{t^2+u^2<\varepsilon^2} (1 - t\,\alpha(s) - u\,\beta(s))\; dt\, du\, ds$$

$$= \int_s \pi\,\varepsilon^2\; ds = L \cdot \pi\,\varepsilon^2$$

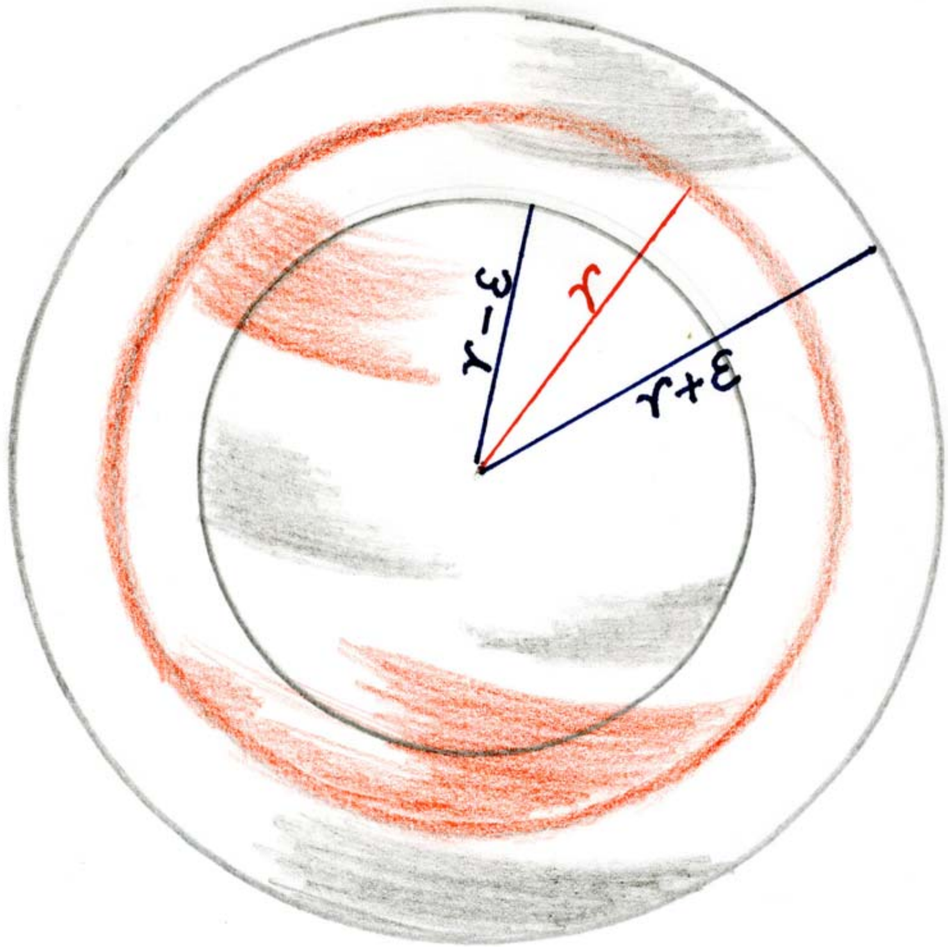
$$= (\text{length of curve})\,(\text{area of } \varepsilon\text{-disk}).$$

Tough Problem. Show that for a smooth curve in R^n , we get

$$\text{vol } \varepsilon\text{-tube} = (\text{length of curve})\,(\text{vol } B^{n-1}(\varepsilon)) ,$$

where $B^{n-1}(\varepsilon)$ is a round ball of radius ε in R^{n-1} .

Tubes about round spheres in 3-space.



The "tube" of radius ε about a round sphere of radius r in 3-space is just the region between the concentric spheres of radii $r + \varepsilon$ and $r - \varepsilon$, with volume

$$4/3\,\pi\,(r + \varepsilon)^3 - 4/3\,\pi\,(r - \varepsilon)^3 = 4/3\,\pi\,(6\,r^2\,\varepsilon + 2\,\varepsilon^3)$$

$$= (4\,\pi\,r^2)\,2\varepsilon + 8/3\,\pi\,\varepsilon^3 = (\text{area of sphere})\,2\varepsilon + 8/3\,\pi\,\varepsilon^3$$

$$= 2\varepsilon\,(\text{area of sphere} + 2\pi/3\,\chi(\text{sphere})\,\varepsilon^2)\,,$$

which is exactly "Weyl's tube formula" for surfaces in R^3.

Problem. Compute the volume of the ε-tube about a torus of revolution in 3-space, and show that it is

$$\text{vol } \varepsilon\text{-tube} = 2\varepsilon \text{ (area of torus)}$$

$$= 2\varepsilon \left(\text{area of torus} + 2\pi/3\, \chi(\text{torus})\, \varepsilon^2\right),$$

since $\chi(\text{torus}) = 0$, again in accord with Weyl's tube formula.