Math 501 - Differential Geometry
Professor Gluck
February 7, 2012

3. THE GEOMETRY OF THE GAUSS MAP

Goal. Learn how surfaces in 3-space are curved.

Outline

Pages 5 - 24. The Gauss map

S = orientable surface in R^3 with choice N of unit normal.

Definition of the *Gauss map* $N: S \to S^2$.

Its differential $dN_p: T_pS \to T_pS^2 = T_pS$.

Self-adjointness of dN_p , meaning

$$< dN_p(V) , W > = < V , dN_p(W) > .$$

Second fundamental form encodes how surface is curved.

Pages 25 - 78. Curvature of curves on a surface

Many examples to develop intuition.

Meusnier's Theorem considers all curves on a given surface through a given point with a given tangent direction there, and compares their curvatures.

Definition of *normal curvature* of a surface at a point in a given direction.

Question. How does the normal curvature of a surface at a point vary as we vary the direction?

Principal curvatures of the surface S at the point p .

Lines of curvature on a surface.

Definition of *Gaussian curvature* and *mean curvature*.

Definition of *umbilical points* on a surface.

Theorem. If all points of a connected surface S are umbilical points, then S is contained in a sphere or a plane.

Pages 79 - 123. The Gauss map in local coordinates

Develop effective methods for computing curvature of surfaces.

Detailed example of a paraboloid.

The *equations of Weingarten* express the entries in the matrix for dN_p in terms of the coefficients of the first and second fundamental forms.

Explicit formulas for principal curvatures, Gaussian and mean curvatures.

Detailed example on a torus of revolution.

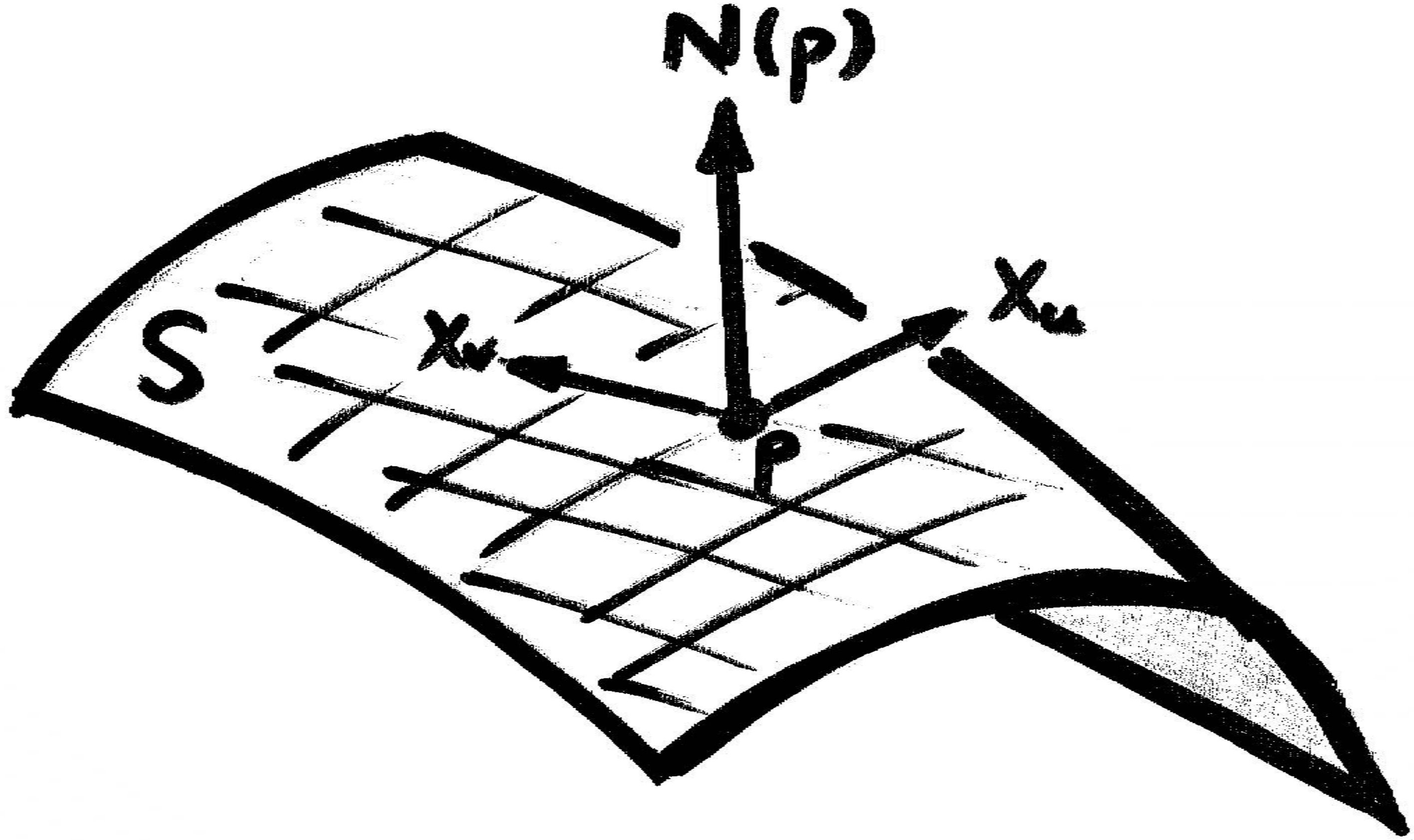
When does a surface lie to one side of its tangent plane at a point?

The Gauss map.

Let $S \subset R^3$ be a regular surface in 3-space, and let $X\colon U \to V \subset S$ be a parametrization of an open set V in S by the open set $U \subset R^2$.

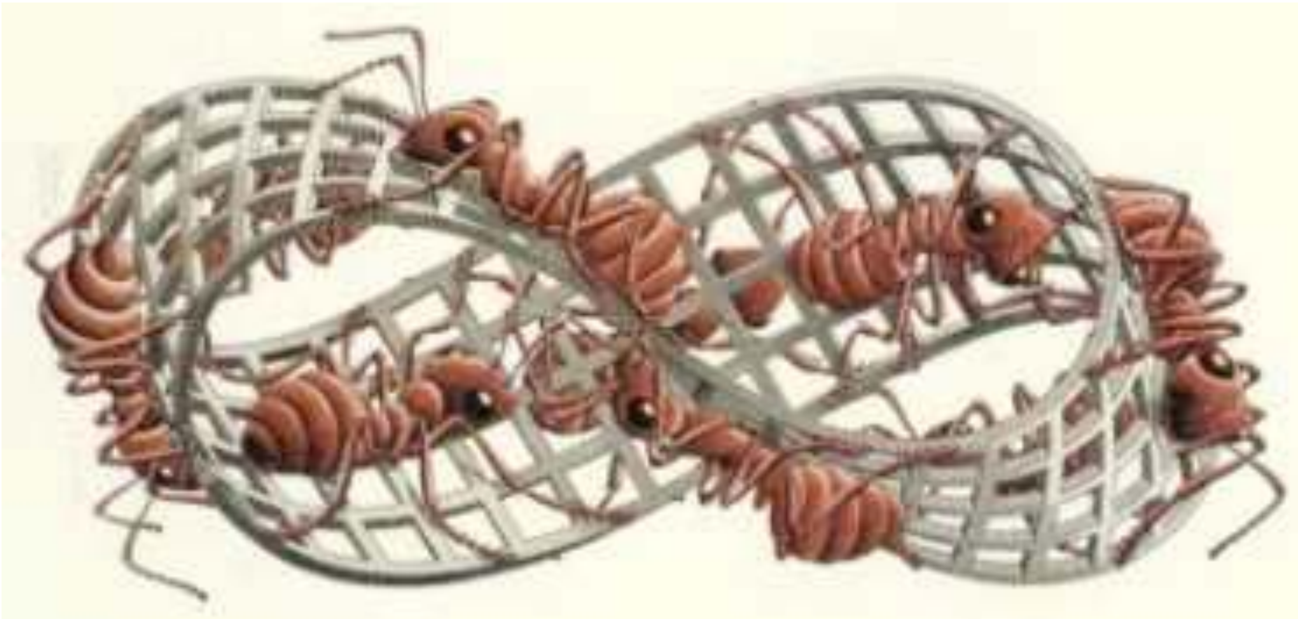
For each point $p \in V$, we can select one of the two possible unit normal vectors to S at p by the rule

$$N(p) = \frac{X_u \times X_v}{|X_u \times X_v|}(p) .$$


N(p)
Xu
Xv
S
p

The map $N: V \rightarrow R^3$ is a differentiable unit normal vector field defined on the open set V in S .

But it may be impossible to define a differentiable unit normal vector field on the whole surface S .



Mobius Band

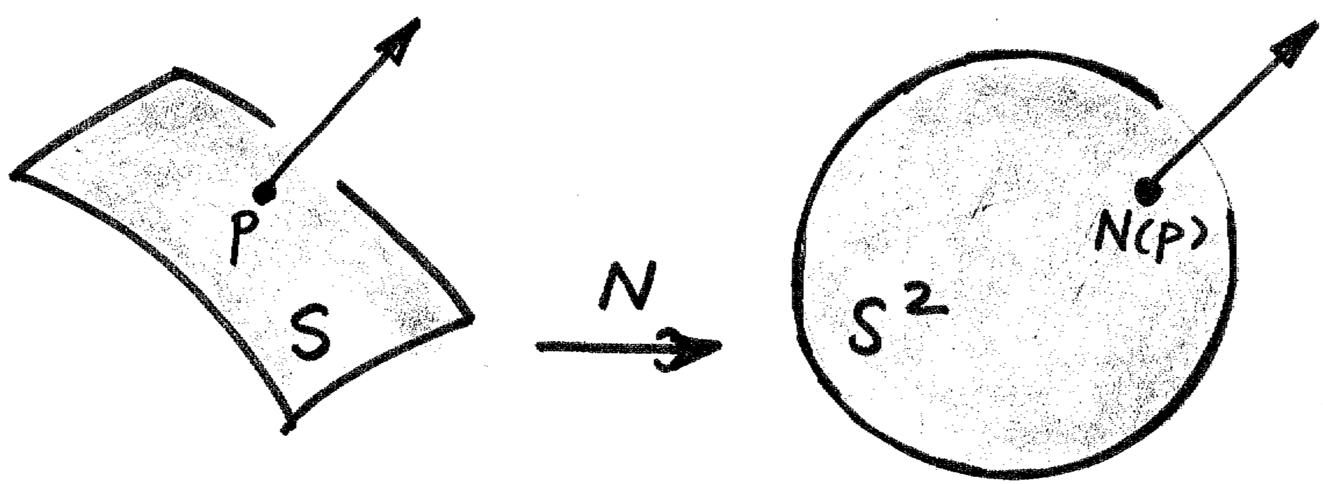
Definition. A regular surface S in R^3 is said to be ***orientable*** if it admits a differentiable field of unit normal vectors on the whole surface S. A choice of such a field is called an ***orientation*** of S.

Problem 1. Show how a choice of an orientation of S serves to orient all the tangent spaces T_pS.

Convention. Throughout this chapter, S will denote a regular orientable surface for which a choice of orientation (that is, a differentiable field N of unit normal vectors) has been made.

Definition. Let $S \subset R^3$ be a regular surface with an orientation N. The map $N: S \to R^3$ takes its values in the unit 2-sphere S^2.

The resulting map $N: S \to S^2$ is called the ***Gauss map*** of the surface S.



The Gauss map is differentiable and its differential at the point $p \in S$, $dN_p: T_pS \to T_{N(p)}S^2$, is a linear map from T_pS to $T_{N(p)}S^2$.

The vector $N(p)$ is normal to S at p and also normal to S^2 at $N(p)$. Hence, as subspaces of R^3, we have $T_pS = T_{N(p)}S^2$. Thus we write

$$dN_p: T_pS \to T_pS .$$

Just as the rate of change of a unit normal vector to a curve in the plane reports the curvature of that curve, the differential dN_p of the Gauss map of a surface S reports information about the curvature of that surface.

Problem 2. Explain and justify the formulas

$$dN_p(X_u) = N_u(p) \quad \text{and} \quad dN_p(X_v) = N_v(p) .$$

Solution. Let $\alpha(t)$ be a curve on the surface S with

$$\alpha(0) = p \quad \text{and} \quad \alpha'(0) = X_u .$$

Then by definition,

$$dN_p(X_u) = (d/dt)|_{t=0} N(\alpha(t)) = N_u(p) .$$

Problem 3. Let $S = S^2(r)$ be a round 2-sphere of radius r about the origin in R^3. Show that for each $p \in S$, the differential

$$dN_p : T_p(S) \to T_p(S)$$

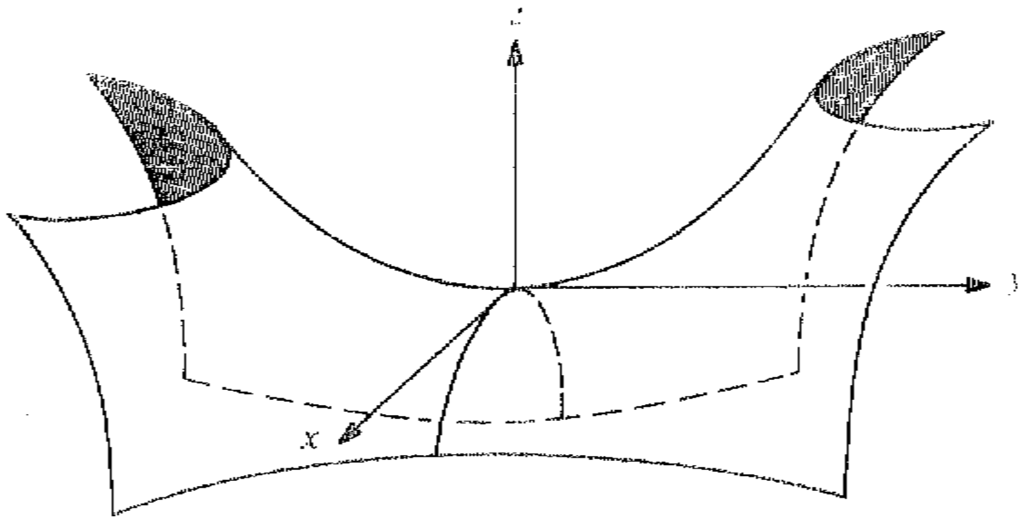
of the Gauss map is given by $dN_p = (1/r)$ Identity .

Problem 4. Let S be the cylinder $x^2 + y^2 = 1$ in R^3. Show that for each $p \in S$, the differential

$$dN_p \colon T_p(S) \rightarrow T_p(S)$$

of the Gauss map is the projection of a vertical 2-plane onto the horizontal line in it.

Problem 5. Let S be the "saddle surface" $z = y^2 - x^2$, known as a ***hyperbolic paraboloid***.



Parametrize S by the map $X : \mathbb{R}^2 \to S \subset \mathbb{R}^3$,

$$X(u, v) = (u , v , v^2 - u^2) .$$

Choose $N(u, v)$ to be the roughly upward pointing unit normal vector to S at the point $X(u, v)$.

(1) Show that

$$X_u = (1, 0, -2u) \quad \text{and} \quad X_v = (0, 1, 2v)$$

$$N(u, v) = (2u , -2v, 1) / (4u^2 + 4v^2 + 1)^{1/2}$$

$$dN_p(X_u) = N_u = (8v^2 + 2, 8uv, -4u) / (4u^2 + 4v^2 + 1)^{3/2}$$

$$dN_p(X_v) = N_v = (8uv, -8u^2 - 2, -4v) / (4u^2 + 4v^2 + 1)^{3/2} .$$

(2) Check that N_u and N_v are both orthogonal to N .

(3) Show that at $p = \text{origin}$, the map $dN_p : T_pS \rightarrow T_pS$ is given by the matrix

$$\begin{matrix} 2 & 0 \\ 0 & -2 \end{matrix}$$

with respect to the basis $X_u = (1, 0, 0)$, $X_v = (0, 1, 0)$ of $T_{origin}S$.

Problem 6. Let S be the quadratic surface $z = ax^2 + by^2$, and as in the previous problem, parametrize S by the map $X\colon R^2 \to S \subset R^3$ given by $X(u, v) = (u, v, au^2 + bv^2)$.

Again let $N(u, v)$ be the roughly upward pointing unit normal vector to S at $X(u, v)$.

Repeat the calculations from the previous problem, and show that at p = origin, the map $dN_p\colon T_pS \to T_pS$ is given by the matrix

$$\begin{matrix} -2a & 0 \\ 0 & -2b \end{matrix}$$

with respect to the basis $X_u = (1, 0, 0)$, $X_v = (0, 1, 0)$ of $T_{origin}S$.

Proposition. *Let* S *be a regular surface in* R^3 . *Then the differential* $dN_p\colon T_pS \to T_pS$ *of the Gauss map at the point* $p \in S$ *is a* ***self-adjoint*** *linear map.*

Proof. If W_1 and $W_2 \in T_pS$, we must show that

$$< dN_p(W_1) , W_2 > \; = \; < W_1 , dN_p(W_2) > .$$

It is sufficient to do this when $W_1 = X_u$ and $W_2 = X_v$ for any parametrization $X(u, v)$ of S near p . Since $dN_p(X_u) = N_u$ and $dN_p(X_v) = N_v$, we must show that

$$< N_u , X_v > \; = \; < X_u , N_v > .$$

Now, N is orthogonal to both X_u and X_v throughout the neighborhood of p on S, so we have

$$< N , X_u > \;=\; 0 \;\text{ and }\; < N , X_v > \;=\; 0 \,.$$

Differentiate the first equation with respect to v and the second with respect to u, getting

$$< N_v , X_u > \;+\; < N , X_{uv} > \;=\; 0 \,, \;\text{ and}$$

$$< N_u , X_v > \;+\; < N , X_{vu} > \;=\; 0 \,.$$

Then $$< N_u , X_v > \;=\; < X_u , N_v >$$

by equality of mixed partials, proving the Proposition.

Problem 7. Let R^n denote n-dimensional Euclidean space with its usual inner product $< , >$.

A linear map $A: R^n \to R^n$ is said to be ***self-adjoint*** if $< A(V), W > \; = \; < V , A(W) >$ for all vectors $V , W \in R^n$.

(a) Prove that A is self-adjoint if and only if its matrix with respect to an orthonormal basis of R^n is symmetric.

(b) Show that if $A: R^n \to R^n$ is a self-adjoint linear map, then the formula $B(V, W) \; = \; < A(V), W >$ defines a symmetric bilinear form $B: R^n \times R^n \to R$, and vice versa.

(c) Let $A: R^n \to R^n$ be self-adjoint, and let V be a unit vector in R^n which maximizes the quantity $< A(V), V >$. Show that V is an eigenvector of A , that is, $A(V) = \lambda V$.

Remark. *Part (c) above is the key step in this problem.*

(d) For this eigenvector V of A , show that if $< V , W > = 0$, then $< V , A(W) > = 0$.

(e) Conclude that if $A: R^n \to R^n$ is a self-adjoint linear map, then there is an orthonormal basis for R^n in terms of which the matrix for A is diagonal.

Now the fact that the differential $dN_p : T_pS \to T_pS$ of the Gauss map is a *self-adjoint* linear map allows us to associate with it a quadratic form Q on T_pS defined by

$$Q(W) = \langle dN_p(W), W \rangle .$$

We can recapture the bilinear form $\langle dN_p(W_1), W_2 \rangle$, and hence the map dN_p itself, by polarizing the quadratic form Q, and hence lose no information by focusing on Q.

Definition. The ***second fundamental form*** of S at p is the quadratic form II_p defined by

$$II_p(W) = -Q(W) = -\langle dN_p(W), W \rangle .$$

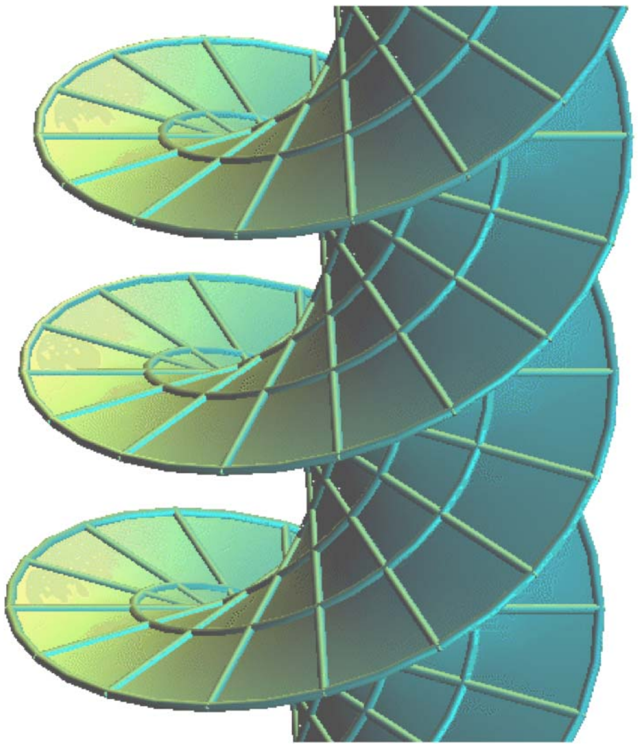
We will shortly explain why we use the minus sign.

Example. At the point $p = (0, 0, 0)$ on the quadratic surface $z = ax^2 + by^2$, the second fundamental form is given by

$$II_p(1, 0, 0) = 2a \quad \text{and} \quad II_p(0, 1, 0) = 2b .$$

Example. Consider the helicoid

$$X(u, v) = (u \cos v , u \sin v , v) .$$



$$X_u = (\cos v , \sin v , 0) \quad \text{and} \quad X_v = (-u \sin v, u \cos v, 1)$$

$$N(u, v) = (\sin v , - \cos v , u) / (1 + u^2)^{1/2}$$

$$N_u = (-u \sin v , u \cos v , 1) / (1 + u^2)^{3/2}$$

$$N_v = (\cos v , \sin v , 0) / (1 + u^2)^{1/2}$$

$$< N_u , X_u > = 0 \qquad < N_u , X_v > = (1 + u^2)^{-1/2}$$

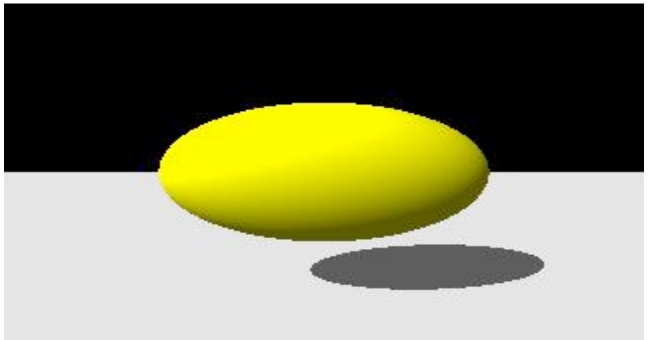
$$< N_v , X_u > = (1 + u^2)^{-1/2} \qquad < N_v , X_v > = 0$$

$$dN(X_u) = N_u = X_v / (1 + u^2)^{3/2}$$

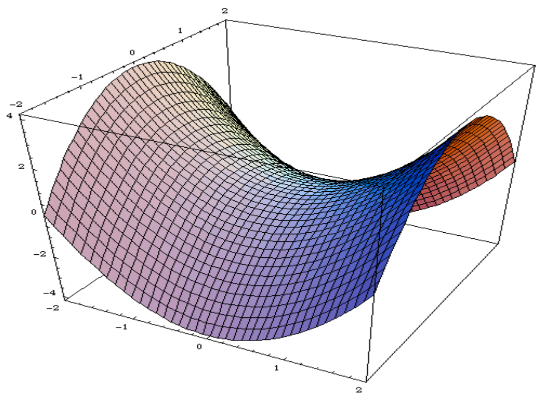
$$dN(X_v) = N_v = X_u / (1 + u^2)^{1/2}$$

Curvature of curves on a surface.

Surfaces can be curved in different amounts,



and in different ways. The ellipsoid above is convex at every point, while the saddle surface below is not.



It is a matter of common sense to try to get a feelingfor the curvature of a surface by investigating the curvature of curves which lie on that surface, and we do this now.

As a quick review, if we are have a regular curve in 3-space defined by $\alpha(t) = (x(t), y(t), z(t))$,

then its curvature is given by $\kappa(t) = |\alpha'(t) \times \alpha''(t)| / |\alpha'(t)|^3$,

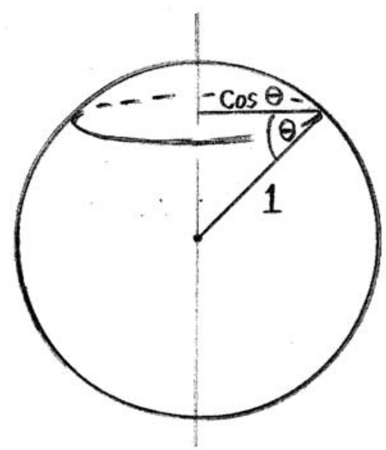
and its principal normal vector $N(t)$ is obtained from

$$\alpha'' - \alpha' (\langle\alpha'', \alpha'\rangle / \langle\alpha', \alpha'\rangle)$$

by dividing this vector by its length.

Problem 8. On a sphere of any radius, consider a great circle and a smaller circle which are tangent to one another at some point. Let κ_G be the curvature of the great circle and κ_S be the curvature of the small circle. Let θ denote the angle between their principal normals at the common point. Show that

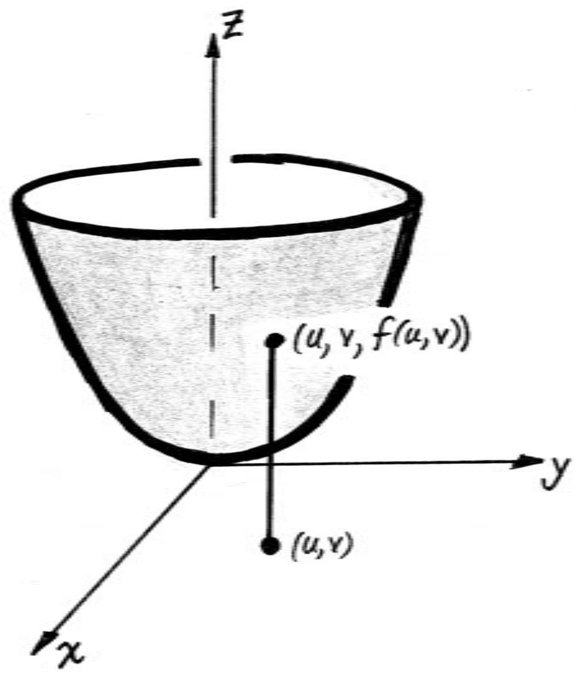
$$\kappa_S = \kappa_G / \cos\theta .$$



Problem 9. Show that the curvature of the parabola $y = \frac{1}{2} a x^2$ at the origin is a.

Given a surface S in 3-space and a point p on S, we can always translate S so as to bring p to the origin, and then rotate S about the origin so that it is tangent to the xy-plane there. Afterwards, near the origin, S is simply the graph of a function $z = f(x, y)$ with

$$f(0, 0) = 0, \quad f_x(0, 0) = 0 \quad \text{and} \quad f_y(0, 0) = 0 .$$



A neighborhood of the origin in the xy-plane can serve as our parameter domain for a neighborhood of the origin on S , with parametrization

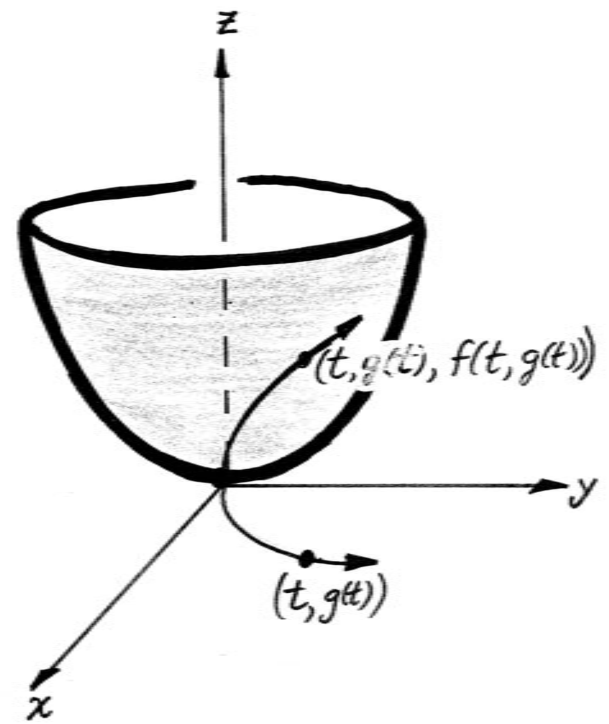
$$X(u, v) = (u , v , f(u, v)) .$$

Let's investigate the curvature of all curves on S which pass through the origin and are tangent there to the x-axis.

In the parameter domain, such a curve is given by

$$u(t) = t \quad \text{and} \quad v(t) = g(t) ,$$

with $g(0) = 0$ and $g'(0) = 0$.



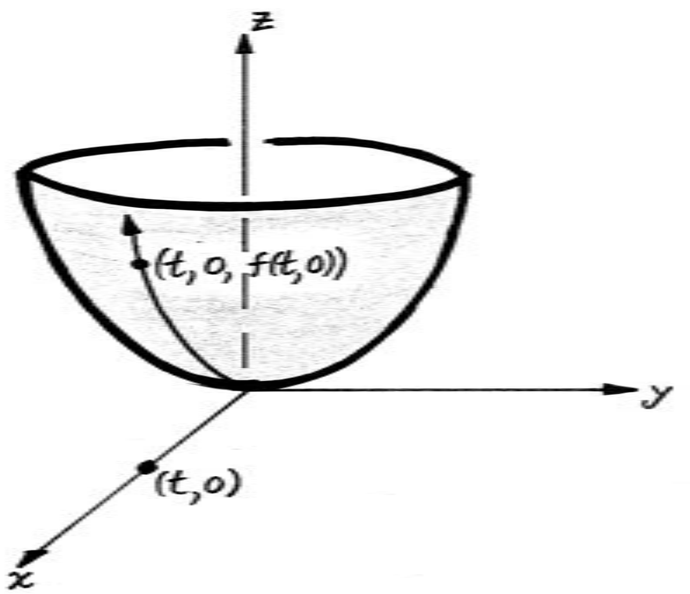
On the surface S , the corresponding curve is

$$\alpha(t) = (t, g(t), f(t, g(t))) .$$

First consider the simplest such curve, say α_0 , which is parametrized by the x-axis itself. Then $g(t) = 0$ and

$$\alpha_0(t) = (t\ ,\ 0\ ,\ f(t, 0))\ .$$

In a neighborhood of the origin, this curve is simply the intersection of the xz-plane with the surface S .



Let's compute the curvature κ_0 at the origin of this curve,

$$\alpha_0(t) = (t\,,\ 0\,,\ f(t, 0))\ .$$

We have

$\alpha_0'(t) = (1\,,\ 0\,,\ f_x(t, 0))$

$\alpha_0'(0) = (1\,,\ 0\,,\ 0)$

$\alpha_0''(t) = (0\,,\ 0\,,\ f_{xx}(t, 0))$

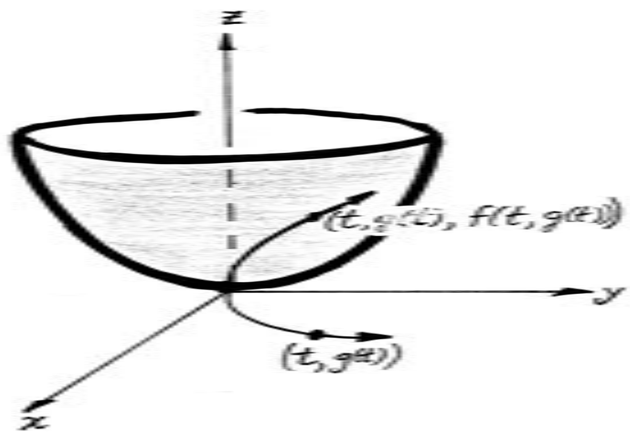
$\alpha_0''(0) = (0\,,\ 0\,,\ f_{xx}(0,0))\ .$

For simplicity of notation, let's write f_{xx} , α_0' , α_0'' and κ_0 and understand that these are evaluated at the origin. Then we get

$$\kappa_0 \;=\; |\alpha_0' \times \alpha_0''| \,/\, |\alpha_0'|^3 \;=\; |(0\,,\; -f_{xx}\,,\; 0)| \;=\; |f_{xx}|\,.$$

Now let's compute the curvature κ_α at the origin of the more general curve

$$\alpha(t) \;=\; (t\,,\; g(t)\,,\; f(t, g(t)))\,.$$



$\alpha'(t) = (1 , g'(t) , f_x + f_y g'(t))$

$\alpha'(0) = (1 , 0 , 0)$

$\alpha''(t) = (0 , g''(t) , f_{xx} + f_{xy}g' + f_{yx}g' + f_{yy}(g')^2 + f_y g'')$

$\alpha''(0) = (0 , g'' , f_{xx})$,

because f_x , f_y and g' are all 0 at the origin.

Then at the origin we get

$$\kappa_\alpha = |\alpha' \times \alpha''| / |\alpha'|^3 = |(0 , -f_{xx} , g'')| = (f_{xx}^2 + g''^2)^{1/2}$$

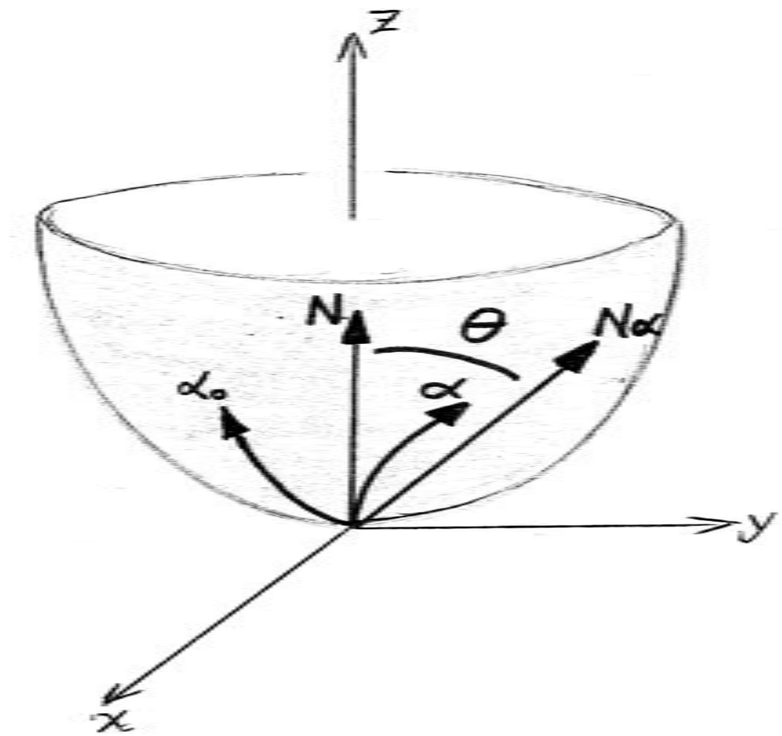
Next we want to compare the curvature κ_α of the general curve $\alpha(t)$ which lies on the surface S and passes through the origin tangent to the x-axis, with the curvature κ_0 of the special curve $\alpha_0(t)$ which is the intersection of the surface S with the xz-plane. Since

$$\alpha' = (1 , 0 , 0) \quad \text{and} \quad \alpha'' = (0 , g'' , f_{xx}) ,$$

the principal normal N_α to the curve α at the origin is

$$N_\alpha = (0 , g'' , f_{xx}) / (g''^2 + f_{xx}^2)^{1/2} .$$

Let θ denote that angle between the unit normal vector $N = (0, 0, 1)$ to the surface at the origin (this is also the principal normal vector to the curve α_0) and the principal normal vector N_α to the curve α at the origin.



Then

$$\cos\theta \;=\; \langle N\,,\,N_\alpha\rangle \;=\; f_{xx}\,/\,(g''^2 + f_{xx}{}^2)^{1/2}\;.$$

Recall that

$$\kappa_\alpha \;=\; (f_{xx}{}^2 \;+\; g''^2)^{1/2} \qquad \text{and} \qquad \kappa_0 \;=\; |f_{xx}|\;.$$

Hence

$$\kappa_\alpha \;=\; \kappa_0\,/\,|\cos\theta|\;.$$

In other words, the curvature of the curve α at the origin depends only on the curvature of the *normal section* α_0 and the angle between the principal normal to α and the normal to the surface.

This is known as ***Meusnier's Theorem.***

We see from Meusnier's Theorem that at the origin on the surface S, the curvature κ_0 of the normal section in the direction of the x-axis is the smallest possible curvature at the origin of any curve on S which is tangent there to the x-axis, since

$$\kappa_\alpha = \kappa_0 / |\cos\theta| .$$

We will call κ_0 the ***normal curvature*** of the surface S at the origin in the direction of the x-axis.

If we agree and fix a unit normal vector N to the surface at that point, then we can even give a sign to the normal curvature κ_0 by orienting the normal plane in a given tangent direction (for example, in the above case, as the xz-plane rather than the zx-plane).

Now we ask, ***how does the normal curvature of a surface at a given point in a given direction vary, as we fix the point but vary the direction?***

As before, we move the surface S so that the point in question is at the origin, and the tangent plane to the surface is the xy-plane. Then our surface has the form

$$X(u, v) = (u, v, f(u, v)),$$

where $f(0, 0) = 0$, $f_u(0, 0) = 0$ and $f_v(0, 0) = 0$.

Looking back at our previous calculations of curvature, we see that only the second derivatives f_{uu}, f_{uv} and f_{vv} at the origin come into play...but no higher derivatives.

Thus we can assume, without loss of generality, that

$$f(u, v) = Au^2 + Buv + Cv^2 .$$

Furthermore, by rotating the surface about the z-axis, we can get rid of the uv term.

Problem 10. Show that rotating the surface about the z-axis to get rid of the uv term above is simply making use of the diagonalizability of a symmetric 2×2 matrix.

So now we can assume, after suitable rotation about the z-axis, that

$$f(u, v) = \tfrac{1}{2} a u^2 + \tfrac{1}{2} b v^2 ,$$

where we use the $\tfrac{1}{2}$ so that $f_{uu} = a$ and $f_{vv} = b$. Thus our surface is parametrized by

$$X(u, v) = (u , v , \tfrac{1}{2} a u^2 + \tfrac{1}{2} b v^2) .$$

Referring back to our previous calculations, we see that the normal curvatures k_1 and k_2 of our surface in the directions of the x- and y-axes are given by

$$k_1 = a \quad \text{and} \quad k_2 = b .$$

Note two things:

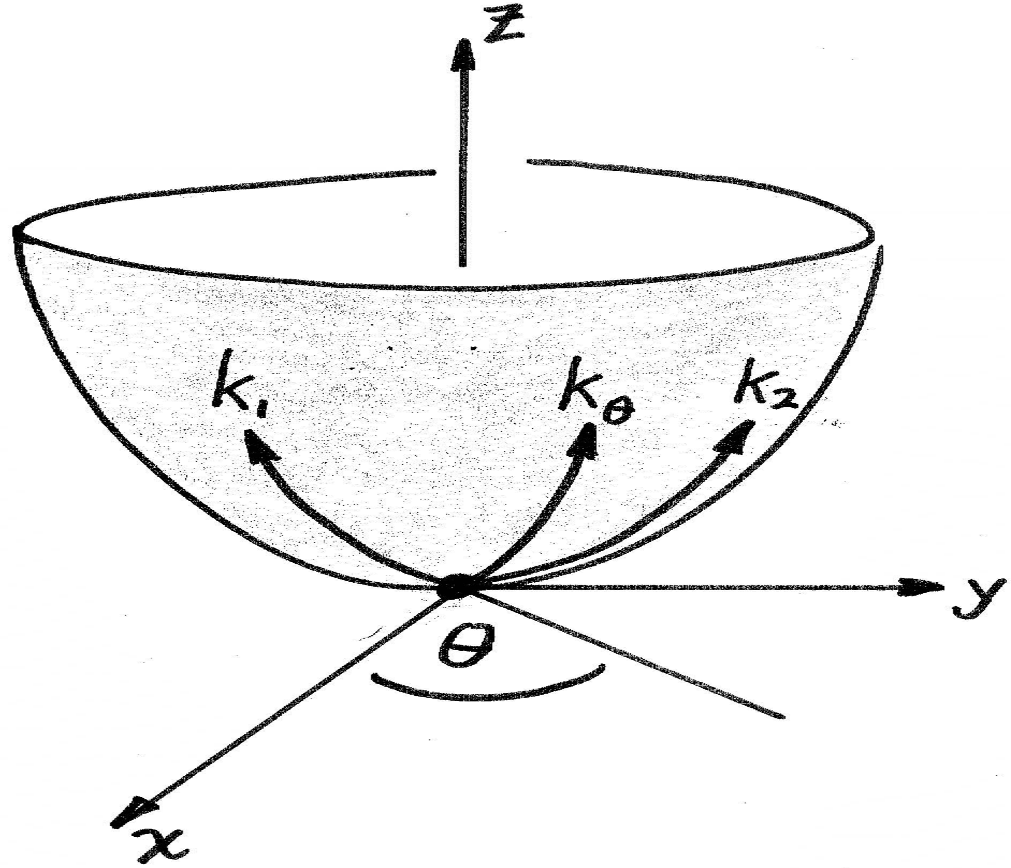
(1) We have switched to do Carmo's notation for normal curvatures, and

(2) We have selected the vector $N = (0, 0, 1)$ as our preferred unit normal to the surface S at the origin, so that normal curvatures now have signs.

Now we want to calculate the normal curvature k_θ of our surface at the origin in the direction of the vector $(\cos\theta , \sin\theta)$ in the xy-plane.

To do this, we consider the curve $\alpha(t)$ on S given by

$$\alpha(t) = (t\cos\theta ,\ t\sin\theta ,\ \tfrac{1}{2} a\, t^2 \cos^2\theta + \tfrac{1}{2} b\, t^2 \sin^2\theta) .$$



$$S = \{(u\,,\ v\,,\ \tfrac{1}{2}\,a\,u^2 + \tfrac{1}{2}\,b\,v^2)\}$$

We calculate.

$$\alpha(t) = (t \cos\theta\, ,\ t \sin\theta\, ,\ \tfrac{1}{2}\, a\, t^2 \cos^2\theta + \tfrac{1}{2}\, b\, t^2 \sin^2\theta)$$

$$\alpha'(t) = (\cos\theta\, ,\ \sin\theta\, ,\ a\, t \cos^2\theta + b\, t \sin^2\theta)$$

$$\alpha'(0) = (\cos\theta\, ,\ \sin\theta\, ,\ 0)$$

$$\alpha''(t) = (0\, ,\ 0\, ,\ a \cos^2\theta + b \sin^2\theta) = \alpha''(0)\, .$$

Since t is an arc-length parameter for our curve α, its signed curvature at the origin, using $N = (0, 0, 1)$ as the preferred normal to the curve, is simply

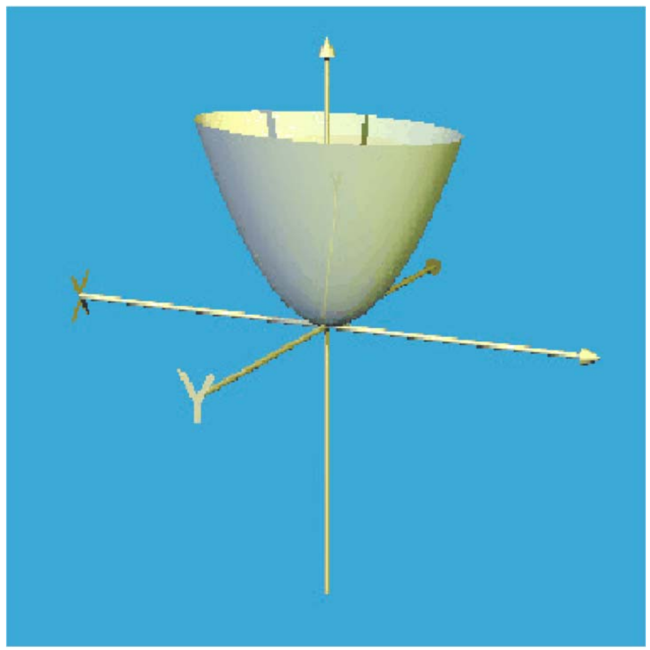
$$k_\alpha = \alpha'' \bullet N = a \cos^2\theta + b \sin^2\theta\, .$$

Then using k_θ in place of k_α, and k_1 and k_2 in place of a and b, we have

$$k_\theta = k_1 \cos^2\theta + k_2 \sin^2\theta ,$$

which tells us how the normal curvature varies with direction.

Definition. The maximum normal curvature k_1 and the minimum normal curvature k_2 are called the ***principal curvatures*** of the surface S at the point p . The corresponding (orthogonal) directions are called the ***principal directions*** at p .



In the previous example, where we considered the point $p = (0, 0, 0)$ on the surface $z = \frac{1}{2} a x^2 + \frac{1}{2} b y^2$, the principal curvatures were the numbers $k_1 = a$ and $k_2 = b$, and the corresponding principal directions were given by the x- and y-axes.

Recall from Problem 6 that in this case we have

$$dN_p(X_u) = -a X_u \quad \text{and} \quad dN_p(X_v) = -b X_v ,$$

so that the principal directions are the directions of the eigenvectors of the differential dN_p of the Gauss map $N: S \to S^2$ at the point p.

Since the second fundamental form II_p was defined by

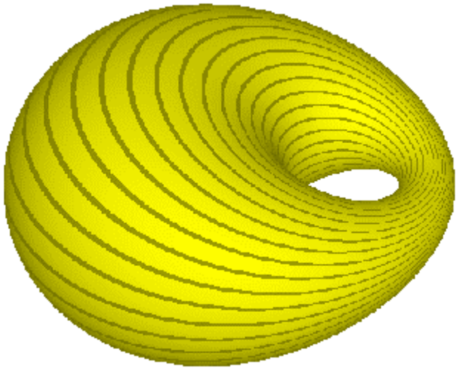
$$II_p(V) = -< dN_p(V) , V > ,$$

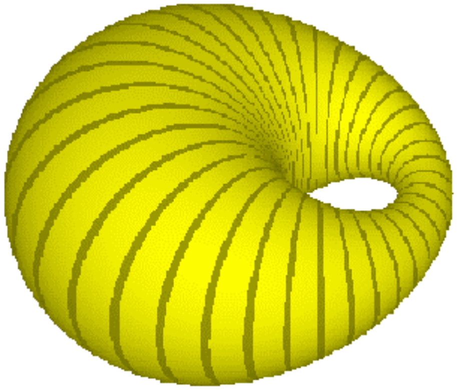
we have

$$II_p(X_u) = -< -a X_u , X_u > = a |X_u|^2$$

$$II_p(X_v) = -< -b X_v , X_v > = b |X_v|^2 .$$

Definition. If a regular connected curve C on the surface S is such that for all points p ϵ C, the tangent line to C at p is a principal direction of S at p, then C is said to be a ***line of curvature*** of S.





Problem 11. Show that a necessary and sufficient condition for a connected regular curve C on S to be a line of curvature is that

$$N'(t) = \lambda(t)\,\alpha'(t)\,,$$

for any parametrization $\alpha(t)$ of C, where $N(t) = N(\alpha(t))$ and $\lambda(t)$ is a differentiable function of t. In this case, $-\lambda(t)$ is the principal curvature along $\alpha(t)$.

Definition. Let S be a regular surface in R^3 and $p \in S$. Let k_1 and k_2 be the principal curvatures of S at p. Then

$$K = k_1 k_2 \quad \text{and} \quad H = \tfrac{1}{2}(k_1 + k_2)$$

are called the ***Gaussian curvature*** and ***mean curvature*** of S at p.

Problem 12. Let $A: R^2 \to R^2$ be a linear map, let V_1, V_2 be a basis for R^2, and let A be the matrix of A with respect to this basis. Show that the determinant and the trace of A do not depend on the choice of basis V_1, V_2, but only on the linear map A. Thus we can refer to these quantities as the determinant and trace of the linear map A.

Problem 13. Recall the Gauss map $N: S \to S^2$ and its differential $dN_p: T_pS \to T_pS^2 = T_pS$.

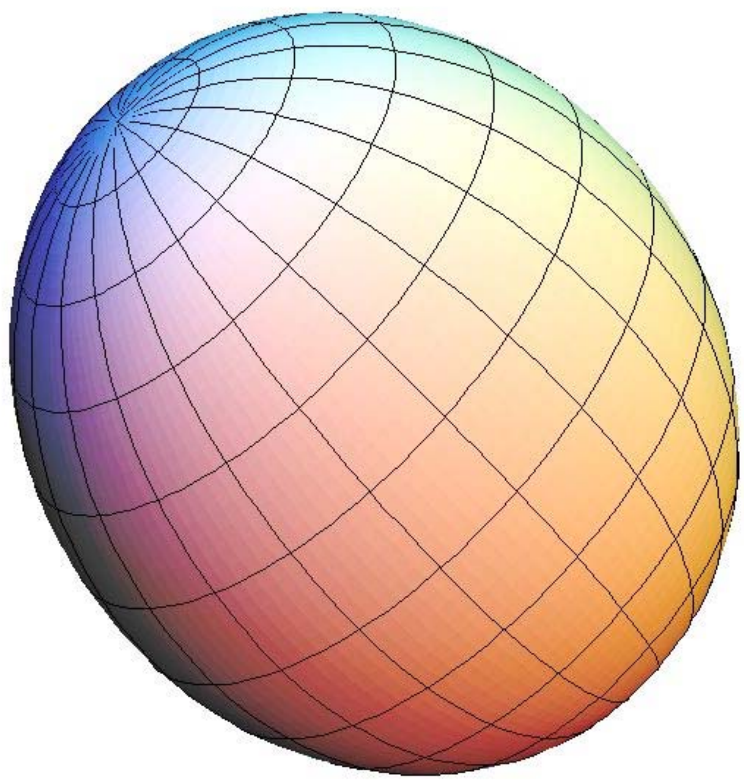
Show that $K = \det dN_p$ and $H = -\frac{1}{2}\,\mathrm{trace}\, dN_p$.

Definition. A point p on a surface S is called

(1) ***elliptic*** if $K > 0$ there (equivalently, the two principal curvatures k_1 and k_2 are either both > 0 or both < 0).

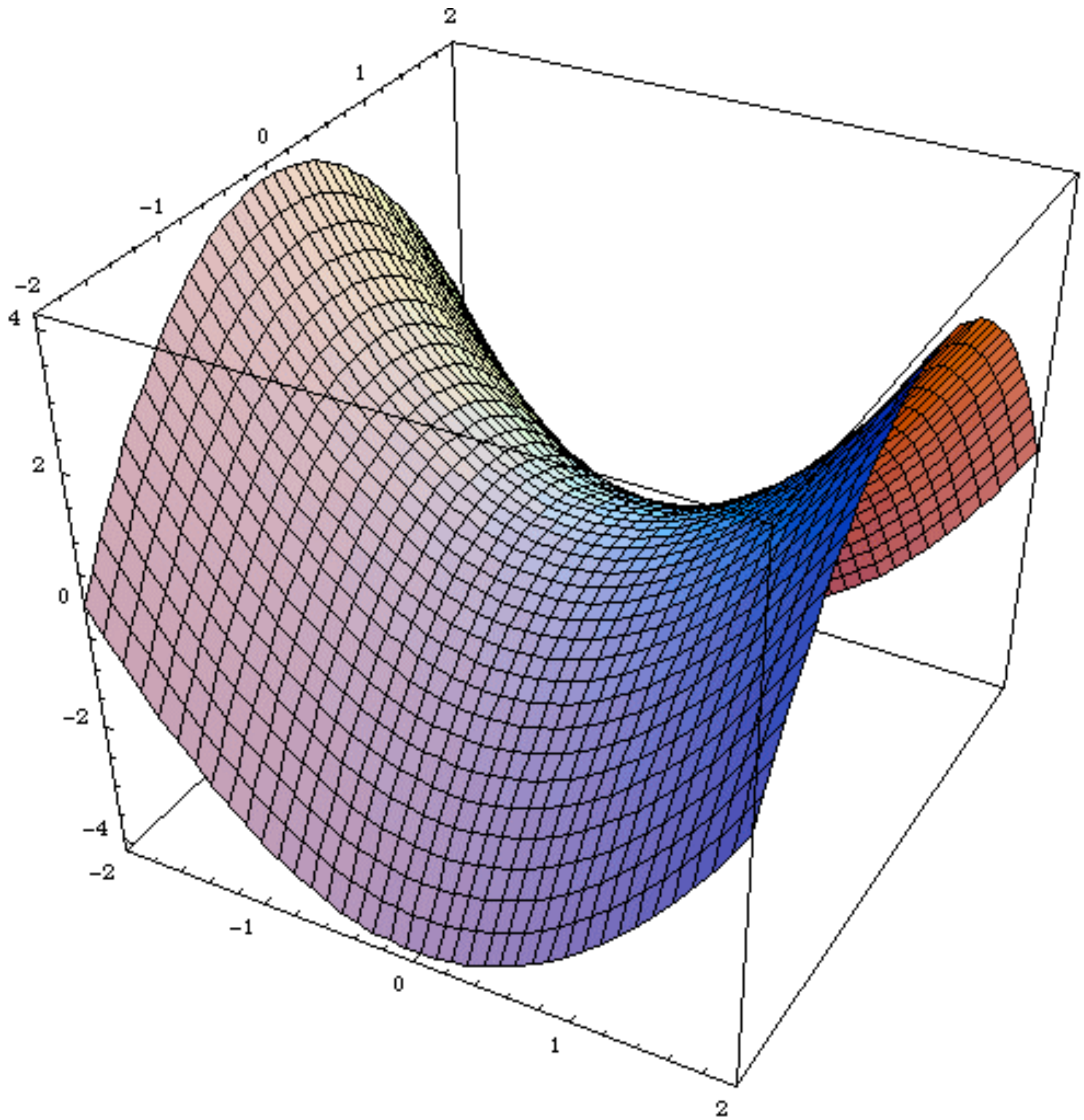
Any point of a sphere or ellipsoid is elliptic.

All curves on S which pass through an elliptic point p have their principal normals pointing toward the same side of the tangent plane T_pS .



(2) ***hyperbolic*** if $K < 0$ there (equivalently, k_1 and k_2 have opposite signs) .

The point $(0, 0, 0)$ of the saddle $z = y^2 - x^2$ is hyperbolic. There are curves on S which pass through a hyperbolic point p which have their principal normals pointing toward either side of T_pS .


2
1
0
-1
-2
4
2
0
-2
-4
-2
-1
0
1
2

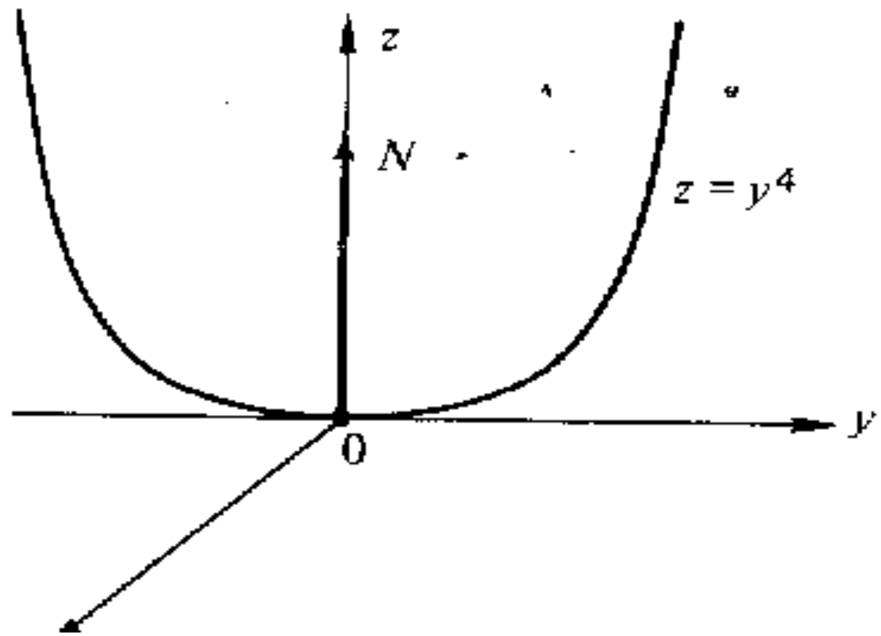
(3) ***parabolic*** if $K = 0$ there, but only one of k_1 and k_2 is 0) .

The points of a circular cylinder are parabolic.

(4) ***planar*** if K = 0 there, and both of k_1 and $k_2 = 0$.

The points of a plane are planar points.

The point (0, 0, 0) is a planar point of the surface
$z = (x^2 + y^2)^2$.



Definition. A point p on a surface S where the two principal curvatures are equal, $k_1 = k_2$, is called an ***umbilical point***. This includes the planar points, where $k_1 = k_2 = 0$.

Examples. All points of a sphere are umbilical points. The point $(0, 0, 0)$ is an umbilical point on the paraboloid $z = x^2 + y^2$.

THEOREM. ***If all points of a connected surface* S *are umbilical points, then* S *is contained in either a sphere or a plane.***

We'll prove this theorem, and begin with some warm-up exercises.

Problem 14.

(a) Prove, as succinctly as possible, that a regular curve in 3-space with zero curvature everywhere is a portion of a straight line.

(b) Prove, again as succinctly as possible, that a regular curve in the plane with constant nonzero curvature is a portion of a circle.

Hint for (b). *Go out from each point of the curve to the point which you believe to be the center of the circle, and then show that this proposed center point does not change as you move along the original curve.*

Problem 15. Prove that if all normal lines to a connected regular surface S meet a fixed straight line L, then S is a portion of a surface of revolution.

Hint. *Go down one dimension and first prove that if all the normal lines to a connected plane curve pass through a fixed point, then that curve is an arc of a circle.*

Then in the original problem, show that the intersection of S with each plane orthogonal to the line L is an arc of a circle, with center on the line L.

Conclude from this that S is a portion of a surface of revolution.

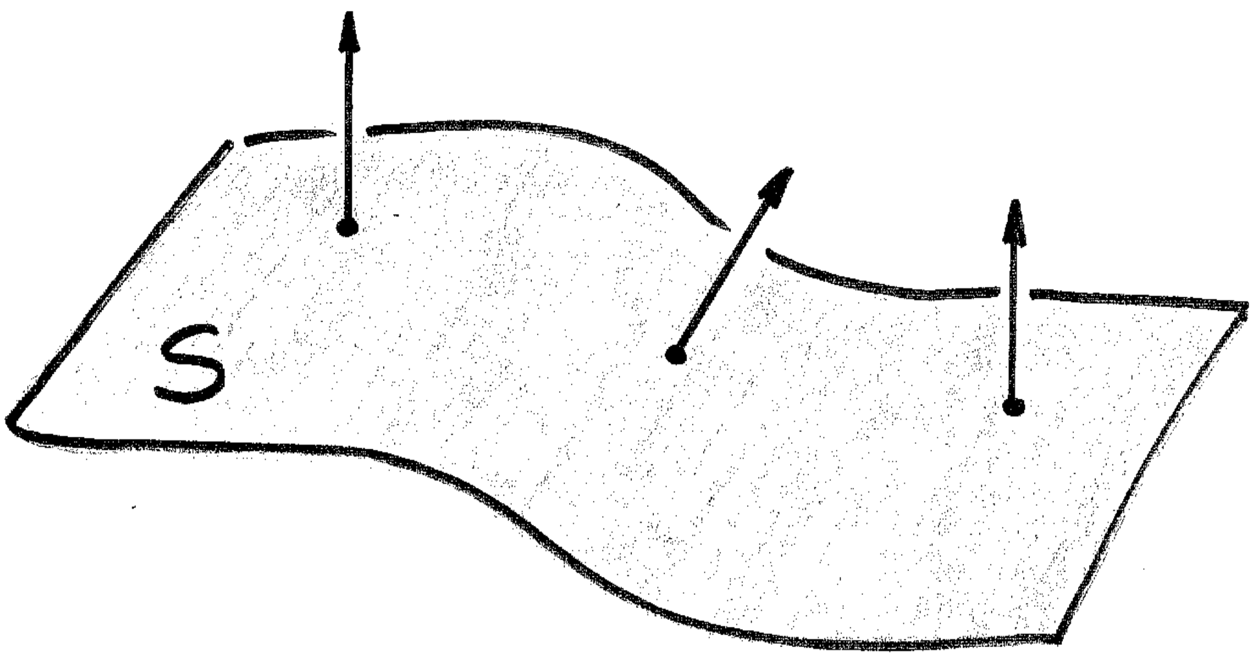
Next, let's consider the special case of the above theorem where we are told that all the points of the connected surface S are planar. We will prove that S is a portion of a plane.

If (u, v) are local coordinates for S in a neighborhood of a point p, then the fact that S is planar at every point tells us that

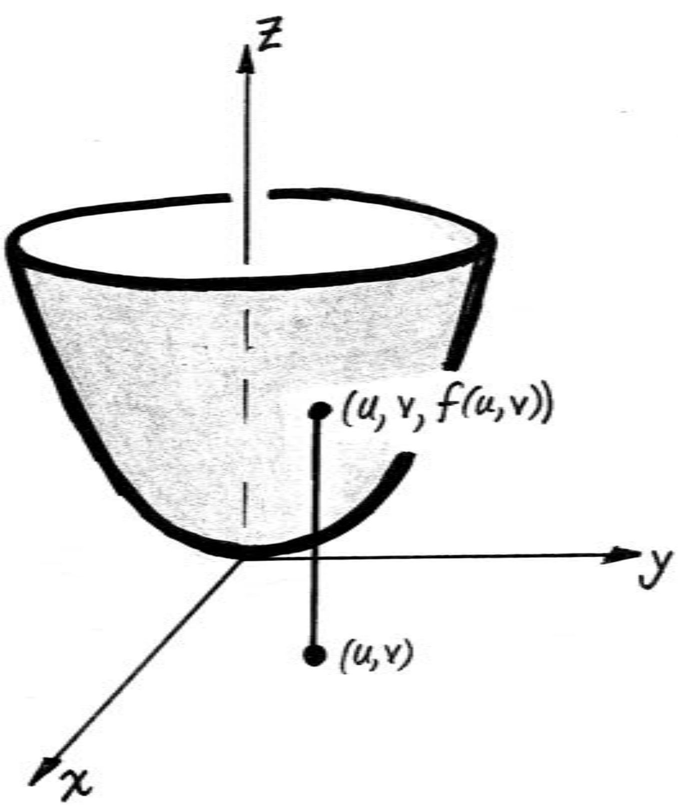
$$dN_p(X_u) = N_u = 0 \quad \text{and} \quad dN_p(X_v) = N_v = 0 ,$$

which tells us that the unit normal vector $N(u, v)$ does not change as u and v change, that is, it is a constant.

It is now at least visually evident that S is a portion of a plane.



To actually prove that S is a portion of a plane, start with any point p on S, then translate and rotate S so as to move p to the origin and make S tangent to the xy-plane there.



Then in these local coordinates we have

$$X(u, v) = (u , v , f(u, v))$$

$$X_u = (1 , 0 , f_x(u, v))$$

$$X_v = (0 , 1 , f_y(u, v))$$

$$X_u \times X_v = (-f_x(u, v) , -f_y(u, v) , 1) .$$

Now at the origin, the unit normal vector N points along the z-axis, say $N = (0, 0, 1)$.

Then it must point in this direction at every point of S , and hence $f_x(u, v) \equiv 0 \equiv f_y(u, v)$.

Thus $f \equiv 0$ in our coordinate neighborhood, so an open neighborhood of p on S lies in a plane.

Since S is connected, all of S lies in the same plane.

We turn now to the general case of our theorem.

Given a connected surface S, all of whose points are umbilical, we must prove that S is contained in either a sphere or a plane.

Given a point p on the surface S, move S as usual so that p goes to the origin and S is tangent to the xy-plane there. Then, near the origin, S is given in local (u, v) coordinates by

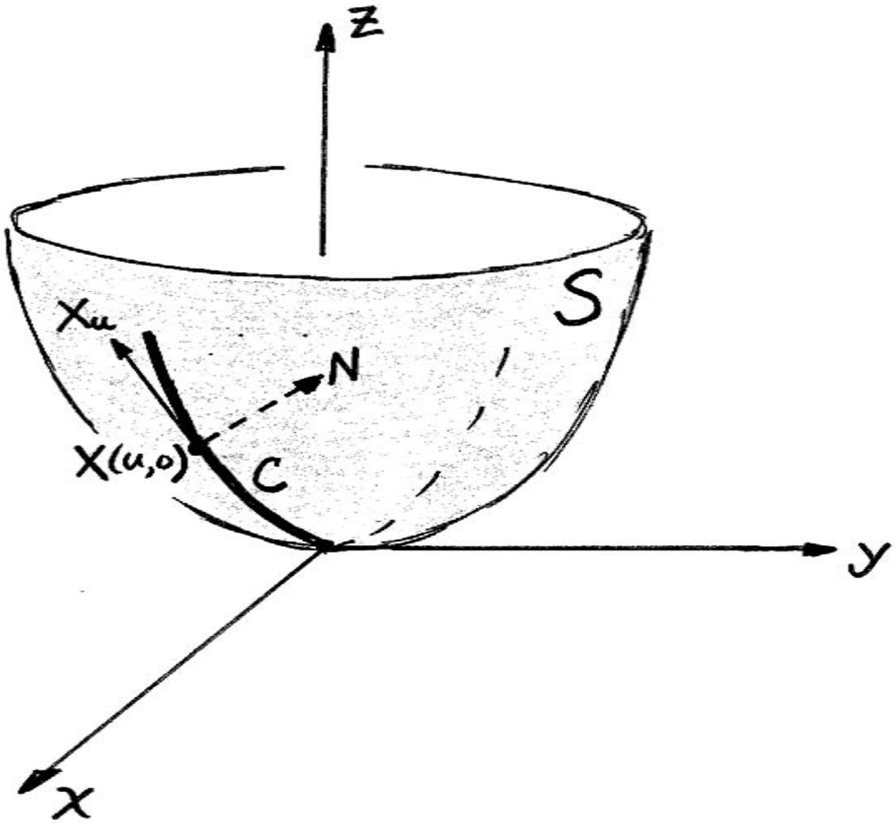
$$X(u, v) = (u, v, f(u, v)).$$

Since each point of S is an umbilical point, let $k(u, v)$ denote the common value of the two principal curvatures of S at the point $X(u, v)$. Then our key equations are

$$dN_p(X_u) = N_u = -k(u, v)\, X_u \ , \quad \text{and}$$

$$dN_p(X_v) = N_v = -k(u, v)\, X_v \ .$$

Focus on the curve C obtained by intersecting our surface S with the xz-plane. In a neighborhood of the origin, the curve C = {X(u, 0)} is parametrized by u .



The tangent vector X_u to C lies in the xz-plane, the unit normal vector N to S at the origin is the vector $(0, 0, 1)$, and the equation

$$N_u = -k(u, 0)\, X_u$$

tells us that the rate of change of N along the curve C also lies in the xz-plane. It follows that the surface normal N itself must lie in the xz-plane along C.

In particular, N serves as a unit normal vector to the curve C.

If $k(0, 0) \neq 0$, then, at least for small u, the line through $X(u, 0)$ spanned by $N(u, 0)$ will intersect the z-axis.

Since we can rotate the surface S about the z-axis to line up any surface direction at the origin with the x-axis, it follows that all the lines normal to the surface S at points near the origin must intersect the z-axis.

By a previous problem, S must be a surface of revolution about the point p.

Since p was an arbitrary point of S, the surface S must, at least locally, be a surface of revolution about each of its points.

Problem 16. Show how it follows from this that the principal curvature $k(u, v)$ must be constant throughout S .

Since we have already dealt with the case that $k \equiv 0$, and shown the corresponding surface S to be a portion of a plane, we will assume now that k is a nonzero constant. By reversing our choice of unit normal vector N to S , we can assume that $k > 0$.

Going back to the curve C , we claim that the principal curvature $k(u, 0) = k$ of the surface S along C is also the ordinary curvature of the plane curve C .

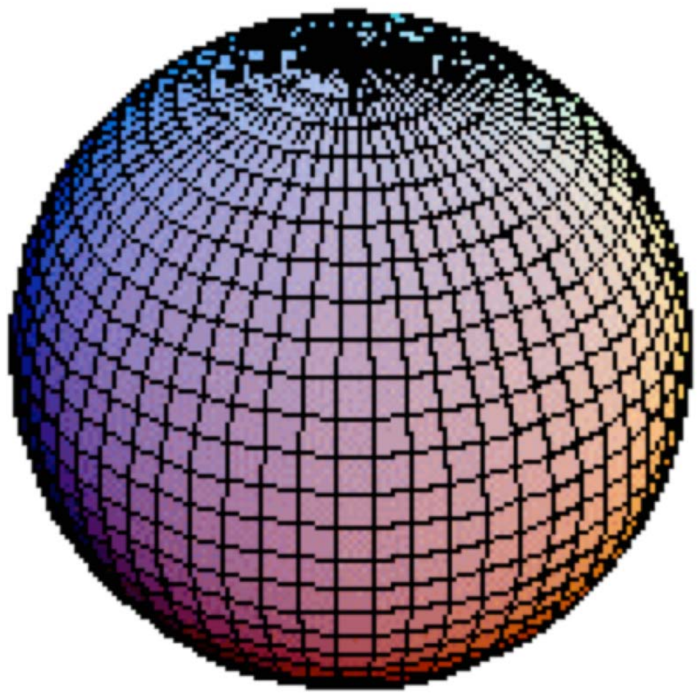
Letting s denote an arc length parameter along C, and recalling that the surface normal $N = N(u, 0)$ is also the principal normal to the curve C, we have

$$dN/ds = (dN/du)\, du/ds = N_u\, du/ds$$

$$= -k\, X_u\, du/ds = -k\, dX/ds ,$$

which, from the Frenet equation, tells us that k is the curvature of the plane curve C.

Thus the curve C lies on a circle of curvature k, and hence the surface S, obtained by rotating C about the z-axis, must lie on a sphere of curvature k, at least in a neighborhood of the origin.



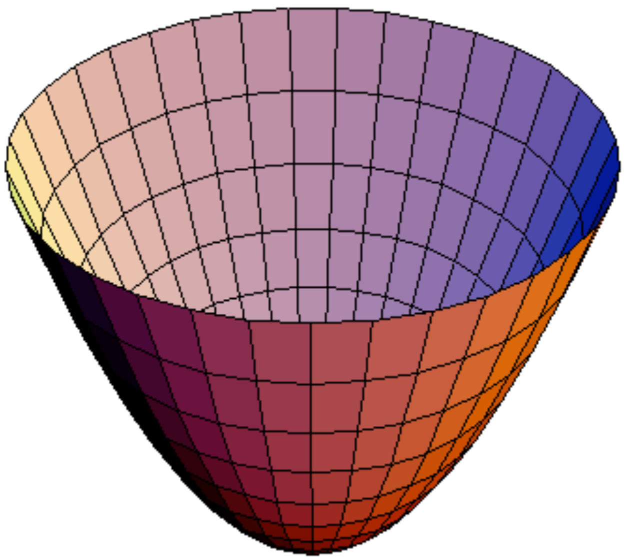
By connectedness, the entire surface S lies on a sphere of curvature k, completing the proof of the theorem.

Remark. Read do Carmo's proof of this theorem, on pages 147 - 148 of his book. I like his proof better.

The Gauss map in local coordinates.

In this section we develop effective methods for computing curvature of surfaces in local coordinates.

To set the tone, we begin with an example.



Circular paraboloid $z = x^2 + y^2$

We can parametrize the circular paraboloid by

$$X(u, v) = (u , v , u^2 + v^2) .$$

Then

$$X_u = (1 , 0 , 2u) \quad \text{and} \quad X_v = (0 , 1 , 2v) .$$

The unit normal vector to S is

$$N(u, v) = X_u \times X_v / |X_u \times X_v|$$

$$= (-2u , -2v , 1) / (1 + 4u^2 + 4v^2)^{1/2} .$$

We then compute

$$N_u \;=\; (-2 - 8v^2\,,\, 8uv\,,\, -4u) \,/\, (1 + 4u^2 + 4v^2)^{3/2}$$

$$N_v \;=\; (8uv\,,\, -2 - 8u^2\,,\, -4v) \,/\, (1 + 4u^2 + 4v^2)^{3/2}\,.$$

Since N_u and N_v lie in the tangent space T_pS, we can express them in the basis X_u and X_v as follows:

$$N_u \;=\; (-2 - 8v^2) \,/\, (\ldots)^{3/2}\; X_u \;\;+\;\; 8uv \,/\, (\ldots)^{3/2}\; X_v$$

$$N_v \;=\; 8uv \,/\, (\ldots)^{3/2}\; X_u \;\;+\;\; (-2 - 8u^2) \,/\, (\ldots)^{3/2}\; X_v\,.$$

Since $N_u = dN_p(X_u)$ and $N_v = dN_p(X_v)$, the four coefficients above give the (transpose of the) matrix for the linear transformation $dN_p\colon T_pS \to T_pS$ in the basis X_u, X_v.

The Gaussian curvature $K = \det dN_p$ and the mean curvature $H = -\frac{1}{2} \operatorname{trace} dN_p$ are then calculated to be

$$K = 4 / (1 + 4u^2 + 4v^2)^2 = 4 / (1 + 4z)^2$$

$$H = (2 + 4z) / (1 + 4z)^{3/2} .$$

The eigenvalues of dN_p are the principal curvatures k_1 and k_2 of S, and the corresponding eigenvectors give the principal directions on S.

They can be calculated from the matrix for dN_p by standard but messy linear algebra.

Now we consider the general case of a surface in R^3 .

Let U be an open set in the plane R^2 and

$$X: U \to S \subset R^3$$

a local parametrization of a portion of a regular surface S in R^3 . We choose

$$N = X_u \times X_v / |X_u \times X_v|$$

as our unit normal vector field throughout the coordinate neighborhood $X(U)$ on S .

Let $\alpha(t) = X(u(t), v(t))$ be a parametrized curve on S, with $\alpha(0) = p$.

The tangent vector to $\alpha(t)$ at p is

$$\alpha' = X_u u' + X_v v',$$

where we understand all quantities to be evaluated at p.

The rate of change of the unit normal vector N to S as we move along the curve α is given by

$$dN(\alpha') = N'(u(t), v(t)) = N_u u' + N_v v' .$$

Since $N_u = dN(X_u)$ and $N_v = dN(X_v)$ both belong to T_pS, we can write

$$N_u = a_{11} X_u + a_{21} X_v \quad \text{and} \quad N_v = a_{12} X_u + a_{22} X_v ,$$

where $a_{ij} = a_{ij}(u, v)$ in our coordinate neighborhood.

Then

$$dN(\alpha') = N_u\, u' + N_v\, v'$$

$$= (a_{11} X_u + a_{21} X_v)\, u' + (a_{12} X_u + a_{22} X_v)\, v'$$

$$= (a_{11} u' + a_{12} v')\, X_u + (a_{21} u' + a_{22} v')\, X_v\, .$$

Thus

$$dN\,(u', v')^T = \begin{matrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{matrix} \quad (u', v')^T$$

This shows that the above matrix (a_{ij}) expresses the linear map $dN_p : T_pS \rightarrow T_pS$ with respect to the basis X_u and X_v .

The expression of the second fundamental form II_p with respect to the basis X_u and X_v for T_pS is given by

$$II_p(\alpha') = -< dN_p(\alpha') , \alpha' >$$

$$= -< N_u u' + N_v v' , X_u u' + X_v v' >$$

$$= e u'^2 + 2f u'v' + g v'^2 , \quad \text{where}$$

$$e = -< N_u , X_u > = < N , X_{uu} >$$

$$f = -<N_v , X_u> = <N , X_{uv}> = <N , X_{vu}> = -<N_u , X_v>$$

$$g = -< N_v , X_v > = < N , X_{vv} > .$$

The quantities $e(u, v)$, $f(u, v)$ and $g(u, v)$ are called the ***coefficients*** of the second fundamental form II in the local (u, v) coordinates.

Example. For the circular paraboloid $z = x^2 + y^2$ discussed above, we have

$$X_u = (1 , 0 , 2u) \quad \text{and} \quad X_v = (0 , 1 , 2v) ,$$

and hence

$$X_{uu} = X_{vv} = (0 , 0 , 2) \quad \text{and} \quad X_{uv} = X_{vu} = (0 , 0 , 0) .$$

We also have

$$N = (-2u , -2v , 1) / (1 + 4u^2 + 4v^2)^{1/2} .$$

Then the coefficients of the second fundamental form are given by

$$e = \langle N , X_{uu} \rangle = 2 / (1 + 4u^2 + 4v^2)^{1/2}$$

$$f = \langle N , X_{uv} \rangle = 0$$

$$g = \langle N , X_{vv} \rangle = 2 / (1 + 4u^2 + 4v^2)^{1/2} .$$

We return to the general case.

Recall that $a_{ij}(u, v)$ are the entries of the matrix for dN_p with respect to the basis X_u and X_v for T_pS .

We next obtain formulas for the functions a_{ij} in terms of the coefficients E , F , G of the first fundamental form and the coefficients e , f , g of the second fundamental form.

Recall that

$$E = \langle X_u , X_u \rangle \qquad F = \langle X_u , X_v \rangle , \qquad G = \langle X_v , X_v \rangle$$

$$e = -\langle N_u , X_u \rangle , \qquad f = -\langle N_u , X_v \rangle = -\langle N_v , X_u \rangle ,$$

$$g = -\langle N_v , X_v \rangle ,$$

$$N_u = a_{11} X_u + a_{21} X_v \quad \text{and} \quad N_v = a_{12} X_u + a_{22} X_v .$$

$$-e = \langle N_u , X_u \rangle = a_{11} E + a_{21} F$$

$$-f = a_{11} F + a_{21} G = a_{12} E + a_{22} F$$

$$-g = a_{12} F + a_{22} G .$$

We can express this in matrix form by

$$-\begin{matrix} e & f \\ f & g \end{matrix} = \begin{matrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{matrix} \quad \begin{matrix} E & F \\ F & G \end{matrix}$$

or compactly by

$$-\,\mathrm{II} = A^t \,\mathrm{I}\,.$$

Thus

$$A^t = -\,\mathrm{II}\;\mathrm{I}^{-1},$$

where

$$\mathrm{I}^{-1} = (EG - F^2)^{-1} \quad \begin{matrix} G & -F \\ -F & E \end{matrix} \quad .$$

We then solve the equation $A^t = - II \; I^{-1}$ for the a_{ij} and get the ***equations of Weingarten***,

$$a_{11} = (f F - e G) / (EG - F^2)$$

$$a_{12} = (g F - f G) / (EG - F^2)$$

$$a_{21} = (e F - f E) / (EG - F^2)$$

$$a_{22} = (f F - g E) / (EG - F^2) .$$

Example. For the circular paraboloid $z = x^2 + y^2$, which serves as our running example, we now use the Weingarten formulas to calculate the entries a_{ij} of the matrix for the linear map $dN_p: T_pS \to T_pS$ with respect to the basis X_u, X_v.

Recall that $$X = (u, v, u^2 + v^2),$$

$$X_u = (1, 0, 2u) \quad \text{and} \quad X_v = (0, 1, 2v).$$

Hence

$E = < X_u , X_u > = 1 + 4u^2$

$F = < X_u , X_v > = 4uv$

$G = < X_v , X_v > = 1 + 4v^2$.

From this, we get

$$EG - F^2 = 1 + 4u^2 + 4v^2 .$$

In the previous example, we calculated

$$e = 2 / (1 + 4u^2 + 4v^2)^{1/2}$$

$$f = 0$$

$$g = 2 / (1 + 4u^2 + 4v^2)^{1/2} .$$

Now we calculate

$$
\begin{aligned}
a_{11} &= (f\,F - e\,G) / (EG - F^2) \\
&= (-2 - 8v^2) / (1 + 4u^2 + 4v^2)^{3/2} \\
a_{12} &= (g\,F - f\,G) / (EG - F^2) \\
&= 8uv / (1 + 4u^2 + 4v^2)^{3/2} \\
a_{21} &= (e\,F - f\,E) / (EG - F^2) \\
&= 8uv / (1 + 4u^2 + 4v^2)^{3/2} \\
a_{22} &= (f\,F - g\,E) / (EG - F^2) \\
&= (-2 - 8u^2) / (1 + 4u^2 + 4v^2)^{3/2} \,.
\end{aligned}
$$

These match the earlier values at the top of page 46, which we used there to calculate the Gaussian and mean curvatures for this example.

As for the Gaussian curvature K in local (u, v) coordinates, we recall the matrix equation

$$-\begin{matrix} e & f \\ f & g \end{matrix} = \begin{matrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{matrix} \quad \begin{matrix} E & F \\ F & G \end{matrix}$$

and immediately get

$$K = a_{11}\, a_{22} - a_{12}\, a_{21} = (eg - f^{\,2}) \,/\, (EG - F^2)\ .$$

If we write the above matrix equation as

$$- II \;=\; A^t\, I \,,$$

then the equation for the Gaussian curvature reads

$$K \;=\; \det II \,/\, \det I \,.$$

Recall that the principal curvatures k_1 and k_2 are the eigenvalues of the linear map $dN_p \colon T_pS \to T_pS$, or equivalently the eigenvalues of the matrix $A = (a_{ij})$ which expresses this linear map in the basis X_u, X_v for T_pS.

Then we defined Gaussian and mean curvatures by

$$K = k_1 k_2 \qquad \text{and} \qquad H = \tfrac{1}{2}(k_1 + k_2) .$$

Thus

$$H = -\tfrac{1}{2}(a_{11} + a_{22})$$

$$= \tfrac{1}{2}(eG - 2fF + gE) / (EG - F^2) .$$

Furthermore, the principal curvatures k_1 and k_2 are roots of the characteristic equation of the matrix A,

$$k^2 - 2Hk + K = 0,$$

and therefore are given by

$$k = H \pm \sqrt{(H^2 - K)}.$$

This equation shows that the principal curvatures are differentiable functions of the coordinates u and v, except possibly at the umbilical points, where $H^2 = K$.

Summary of formulas. We start with the local parametrization $X: U \to S$ of a surface S in R^3.

We use the basis X_u, X_v for the tangent plane T_pS, and write a general tangent vector in T_pS as the velocity vector

$$\alpha' = X_u u' + X_v v'$$

of a curve $\alpha(t) = X(u(t), v(t))$ on S.

The unit normal vector field $N = X_u \times X_v / |X_u \times X_v|$ to S leads to the Gauss map $N: S \to S^2$, whose differential $dN_p : T_pS \to T_pS^2 = T_pS$ has the matrix $A = (a_{ij})$ in the basis X_u, X_v for T_pS.

The first fundamental form is defined to be

$$I(\alpha') = < \alpha' , \alpha' > = < X_u u' + X_v v' , X_u u' + X_v v' >$$
$$= E u'^2 + 2F u' v' + G v'^2 , \quad \text{with}$$

$$E = < X_u , X_u > \qquad F = < X_u , X_v > , \qquad G = < X_v , X_v > .$$

The second fundamental form is defined to be

$$II(\alpha') = - < dN_p(\alpha') , \alpha' >$$
$$= - < N_u u' + N_v v' , X_u u' + X_v v' >$$
$$= e u'^2 + 2f u' v' + g v'^2 , \quad \text{with}$$

$$e = - < N_u , X_u > = < N , X_{uu} >$$

$$f = - < N_u , X_v > = - < N_v , X_u > = < N , X_{uv} >$$

$$g = - < N_v , X_v > = < N , X_{vv} > .$$

The entries a_{ij} of the matrix A which represents $dN_p : T_pS \rightarrow T_pS$ in the basis X_u , X_v are given by the Weingarten equations

$$a_{11} = (f\,F - e\,G) / (EG - F^2)$$

$$a_{12} = (g\,F - f\,G) / (EG - F^2)$$

$$a_{21} = (e\,F - f\,E) / (EG - F^2)$$

$$a_{22} = (f\,F - g\,E) / (EG - F^2)\ .$$

The Gaussian curvature K is given by

$$K = a_{11} a_{22} - a_{12} a_{21} = (eg - f^2) / (EG - F^2) .$$

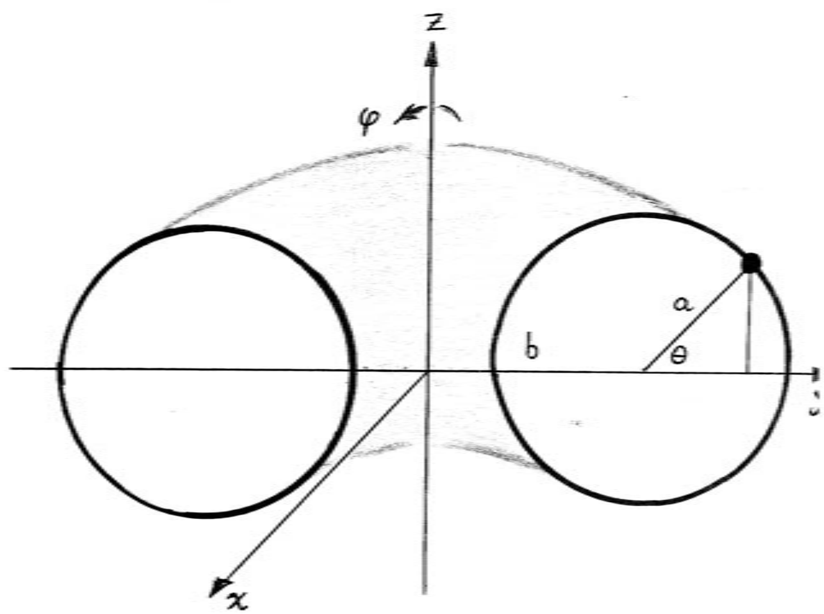
The mean curvature H is given by

$$H = -\tfrac{1}{2}(a_{11} + a_{22}) = \tfrac{1}{2} (eG - 2 f F + gE) / (EG - F^2) .$$

The principal curvatures k_1 and k_2 are given by

$$k_1 , k_2 = H \pm \sqrt{(H^2 - K)} .$$

Example. Consider the torus of revolution pictured below.



$$X(\theta, \varphi) = ((b + a\cos\theta)\cos\varphi\,,\ (b + a\cos\theta)\sin\varphi\,,\ a\sin\theta)$$

First we compute a basis for the tangent space T_pS .

$$X_\theta = (-\,a\sin\theta\cos\varphi\,,\ -\,a\sin\theta\sin\varphi\,,\ a\cos\theta)$$

$$X_\varphi = (-(b + a\cos\theta)\sin\varphi\,,\ (b + a\cos\theta)\cos\varphi\,,\ 0)$$

Then we compute the coefficients of the first fundamental form $I = E\,u'^2 + 2F\,u'v' + G\,v'^2$.

$$E = \langle X_\theta , X_\theta \rangle = a^2$$

$$F = \langle X_\theta , X_\varphi \rangle = 0$$

$$G = \langle X_\varphi , X_\varphi \rangle = (b + a\cos\theta)^2$$

$$EG - F^2 = a^2 (b + a\cos\theta)^2$$

Next we compute the unit outward normal and its derivatives.

$X_\varphi \times X_\theta \; = \; a\,(b + a\cos\theta)\,(\cos\theta\cos\varphi\,,\,\cos\theta\sin\varphi\,,\,\sin\theta)$

$N \; = \; X_\varphi \times X_\theta \,/\, |\, X_\varphi \times X_\theta| \; = \; (\cos\theta\cos\varphi\,,\,\cos\theta\sin\varphi\,,\,\sin\theta)$

$N_\theta \; = \; (-\sin\theta\cos\varphi\,,\,-\sin\theta\sin\varphi\,,\,\cos\theta)$

$N_\varphi \; = \; (-\cos\theta\sin\varphi\,,\,\cos\theta\cos\varphi\,,\,0)$

Then we compute the coefficients of the second fundamental form $II \; = \; e\,u'^2 \; + \; 2f\,u'v' \; + \; g\,v'^2$.

$e \; = \; -< N_\theta\,,\,X_\theta > \; = \; -a$

$f \; = \; -< N_\theta\,,\,X_\varphi > \; = \; 0$

$g \; = \; -< N_\varphi\,,\,X_\varphi > \; = \; -(b + a\cos\theta)\cos\theta$

Next we compute the entries a_{ij} of the matrix A which represents $dN_p: T_pS \rightarrow T_pS$ in the basis X_u , X_v .

$$a_{11} = (f\,F - e\,G) / (EG - F^2) = 1/a$$

$$a_{12} = (g\,F - f\,G) / (EG - F^2) = 0$$

$$a_{21} = (e\,F - f\,E) / (EG - F^2) = 0$$

$$a_{22} = (f\,F - g\,E) / (EG - F^2) = \cos\theta / (b + a\cos\theta)$$

Next we compute the Gaussian and mean curvatures.

$$K = (eg - f^2) / (EG - F^2) = \cos\theta / (a (b + a \cos\theta))$$

$$H = \tfrac{1}{2} (eG - 2 f F + gE) / (EG - F^2)$$
$$= - \tfrac{1}{2} (b + 2a \cos\theta) / (a (b + a \cos\theta))$$

Then we compute the principal curvatures.

$$H^2 - K = \tfrac{1}{4} b^2 / (a^2 (b + a \cos\theta)^2)$$

$$\sqrt{(H^2 - K)} = \tfrac{1}{2} b / (a (b + a \cos\theta))$$

$$k_1 = H - \sqrt{(H^2 - K)} = - 1/a$$

$$k_2 = H + \sqrt{(H^2 - K)} = - \cos\theta / (b + a \cos\theta)$$

From the formula for the Gaussian curvature,

$$K = \cos\theta / (a (b + a \cos\theta)) ,$$

we see that $K > 0$ when $-\pi/2 < \theta < \pi/2$, that is, on the outside of the torus, and that $K < 0$ when $\pi/2 < \theta < 3\pi/2$, that is, on the inside on the torus.

We see that $K = 0$ along the top and bottom circles, where $\theta = \pi/2$ and where $\theta = 3\pi/2$ "=" $-\pi/2$.

The outside points are elliptic, the inside points are hyperbolic, and the top and bottom circles consist of parabolic points.

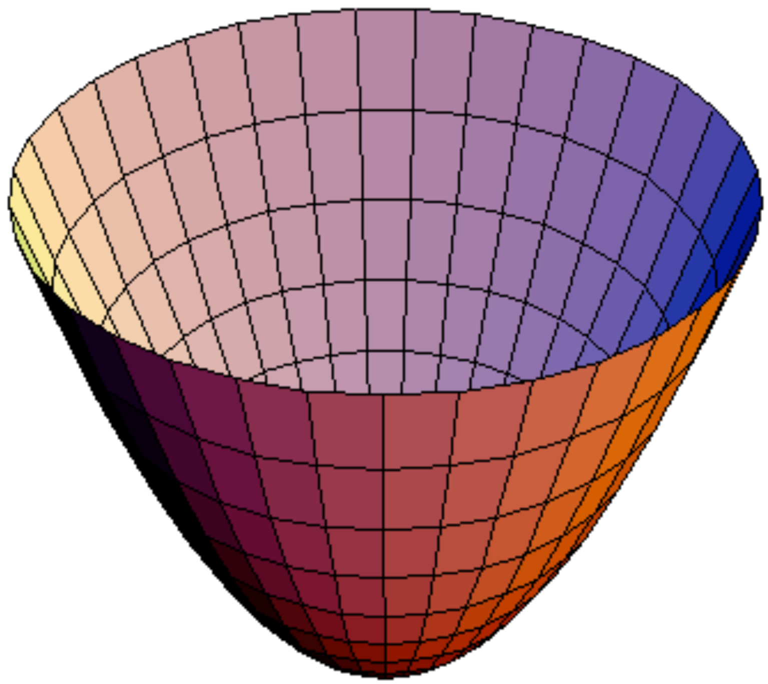
When $\theta = 0$ we are on the outermost circle of the torus, where $K = (1/a)(1/(a + b))$. Here, the outermost circle of radius $a + b$ and the meridian circle φ = constant, of radius a, are both normal sections of the torus in principal directions.

Hence the Gaussian curvature there is the product of the curvatures of these two circles.

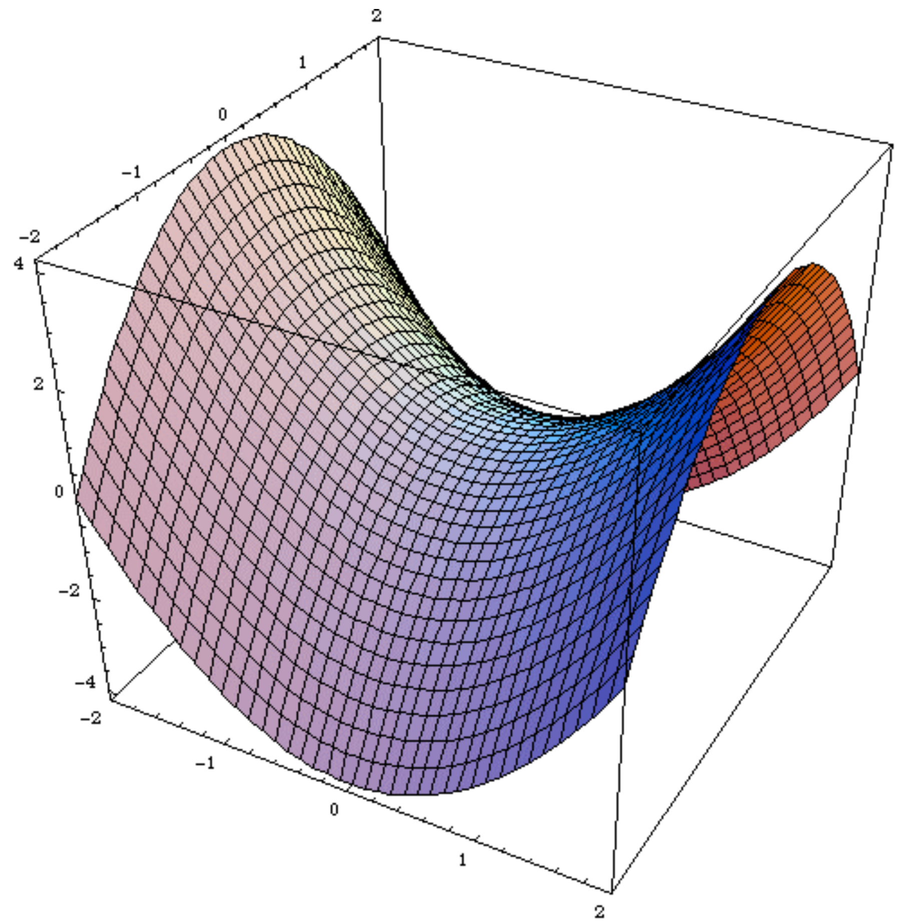
When $\theta = \pi$ we are on the very innermost circle of the torus, where $K = (-1/a)(1/(b-a))$. Here, the innermost circle has radius $b-a$ and the meridian circle has radius a. Again they are both normal sections of the torus in principal directions.

The negative sign of the curvature is due to the fact that the innermost circle curves toward the outer normal, while the meridian circle curves away from it.

Proposition. *Let* $p \in S$ *be an elliptic point on the regular surface* S *. Then there is a neighborhood* V *of* p *on* S *such that all points in* V *, other than* p *itself, lie in the same open half space determined by the tangent plane* T_pS *.*



If p *is a hyperbolic point, then every neighborhood* V *of* p *on* S *contains points lying in both open half spaces determined by* T_pS .



Before proving this, we warm up with the following

Problem 17. Consider the graph $y = f(x)$ of a smooth function f such that $f(0) = 0$ and $f'(0) = 0$. Then the graph goes through the origin and is tangent there to the x-axis. Suppose that $f''(0) > 0$. Then there is a neighborhood V of the origin on the graph of f such that all points of V, other than the origin itself, lie in the open upper half plane of the xy-plane.

Hint. Use Taylor's theorem which says that

$$f(x) = f(0) + f'(0)\,x + \tfrac{1}{2} f''(0)\,x^2 + R(x),$$

where the remainder $R(x)$ satisfies $\mathrm{Lim}_{x \to 0} |R(x)| / x^2 = 0$.

Proof of Proposition. Let p be an elliptic point on the regular surface S. As usual, we can take p to be the origin, and S to be tangent there to the xy-plane. Then a neighborhood V of p on S is the graph of a function $f : U \to R$, where U is a neighorhood of the origin in the xy-plane:

$$V = \{X(u, v) = (u, v, f(u, v)) : (u, v) \in U\} .$$

We have $f(0, 0) = 0$, $f_x(0, 0) = 0$ and $f_y(0, 0) = 0$. Therefore

$$f(u, v) = \tfrac{1}{2}(f_{xx}(0,0)\, u^2 + 2\, f_{xy}(0,0)\, uv + f_{yy}(0,0)\, v^2) + R(u,v) ,$$

and $$\lim_{(u,v)\to(0,0)} R(u, v) / (u^2 + v^2) = 0 .$$

Notice that at the origin we have

$e = < N , X_{uu} > = f_{xx}(0, 0)$

$f = < N , X_{uv} > = f_{xy}(0, 0)$

$g = < N , X_{vv} > = f_{yy}(0, 0)$.

Therefore we can write

$$f(u, v) = ½(e\,u^2 + 2\,f\,uv + g\,v^2) + R(u, v) .$$

Since

$$K = (eg - f^2) / (EG - F^2) ,$$

and $EG - F^2 > 0$, we see that the sign of K agrees with the sign of $eg - f^2$.

At an elliptic point, we have $eg - f^2 > 0$, and it follows easily that there is a real number $a > 0$ so that

$$e\,u^2 + 2\,f\,uv + g\,v^2 > a\,(u^2 + v^2)\,.$$

If we choose our neighborhood U of $(0, 0)$ in the xy-plane so small that $R(u, v) < \frac{1}{2}\,a\,(u^2 + v^2)$ within that neighborhood, that we will have $f(u, v) > 0$ for $(u, v) \in U$.

It follows that the surface S, which is the graph of f, lies above the xy-plane for all $(u, v) \in U$, as claimed.

If p is a hyperbolic point, then normal cross sections of S in the two principal directions easily provide points which lie on both sides of the tangent plane T_pS.

This completes the proof of the proposition.

Example. The "monkey saddle"

$$X(u, v) = (u\,,\; v\,,\; u^3 - 3\,v^2\,u)$$

is shown below.

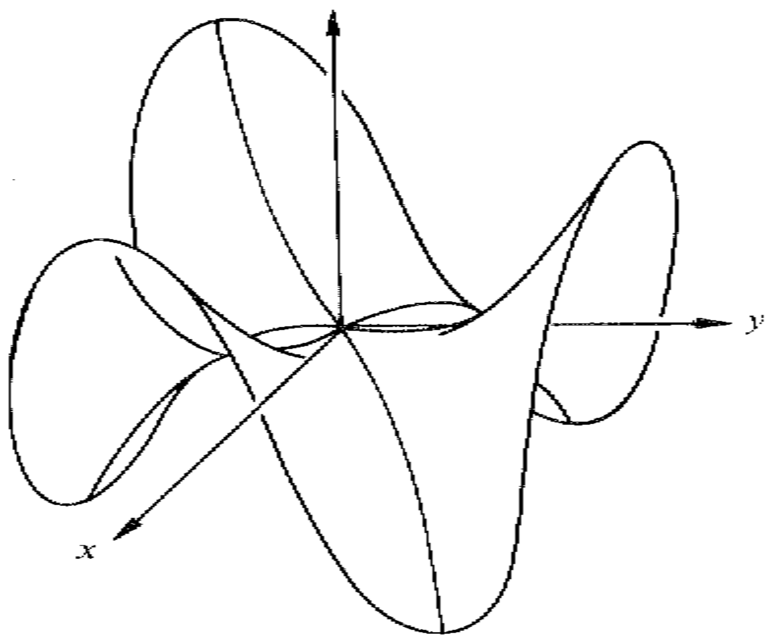


Figure 3-17

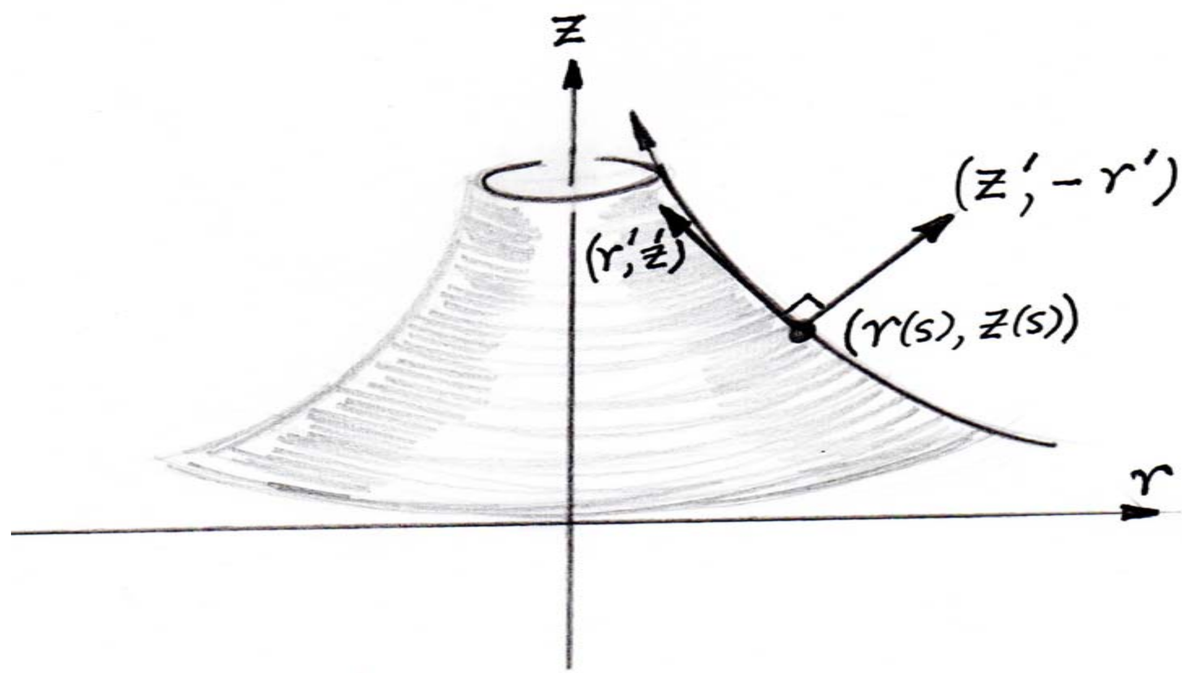
The coefficients e , f , g of the second fundamental form are all zero at the origin, so the origin is a planar point on this surface.

The xy-plane is the tangent plane to this surface at the origin.

Every neighborhood of the origin on this surface has points which lie above, and also points which lie below, the tangent plane.

Problem 18. Let S be a surface of revolution in 3-space. Calculate the first and second fundamental forms, the Gaussian, mean and principal curvatures, and the principal directions at all points of S .

Solution. We sketch below the curve in the rz - plane which we then rotate about the z-axis to get the surface S .



We assume our curve in the rz - plane is parametrized by arc length s, and write

$$X = (r(s) \cos \varphi , r(s) \sin \varphi , z(s))$$

$$X_s = (r' \cos \varphi , r' \sin \varphi , z')$$

$$X_\varphi = (-r \sin \varphi , r \cos \varphi , 0) ,$$

note happily that $X_s \perp X_\varphi$, and calculate that

$$N = (z' \cos \varphi , z' \sin \varphi , -r') .$$

We check that N is orthogonal to both X_s and X_φ, and that it is of unit length because s is an arc length parameter.

We continue...

$$dN(X_s) = N_s = (z'' \cos\varphi , z'' \sin\varphi , -r'')$$

$$dN(X_\varphi) = N_\varphi = (-z' \sin\varphi , z' \cos\varphi , 0) ,$$

and check that N_s and N_φ are both orthogonal to N .
We next compute that

$$N_s = (r' z'' - r'' z') X_s$$

$$N_\varphi = (z' / r) X_\varphi .$$

Hence the matrix for the linear map $dN_p : T_pS \rightarrow T_pS$ is the diagonal matrix

$$\mathrm{diag}(r' z'' - r'' z' , z' / r) .$$

Conclusions so far:

The tangent vectors X_s and X_φ give the *principal directions*.

The corresp. *principal curvatures* are $r' z'' - r'' z'$ and z' / r .

Gaussian curvature is their product:

$$K = (r' z'' - r'' z')(z' / r) .$$

Mean curvature is their average

$$H = 1/2\, (r' z'' - r'' z' + z' / r) .$$

Continuing...

The principal curvature $r'z'' - r''z'$ is exactly the curvature of the curve in the rz - plane which was rotated about the z-axis to produce the surface S .

The principal curvature z'/r is the curvature $1/r$ of the horizontal circle on S , multiplied by z' , which is the cosine of the angle between the principal normal to this circle and the normal to the surface S .

Quick check.

Consider the circle of radius R is the rz - plane given in arc length parametrization by

$$r = R\cos(s/R) \quad \text{and} \quad z = R\sin(s/R) .$$

If we rotate this about the z-axis, we get a round sphere of radius R .

We use the above formulas for principal curvatures and see that both in this case equal $1/R$, as they should.

Problem 19. Let the surface S be given as the graph

$$X(u, v) = (u, v, f(u, v)).$$

Calculate the first and second fundamental forms, the Gaussian, mean and principal curvatures, and the principal directions at all points of S.

Hint. Many satisfying cancellations lead to the formula

$$K = \frac{f_{uu} f_{vv} - f_{uv}^2}{(1 + f_u^2 + f_v^2)^2}$$

for the Gaussian curvature. Confirm this for a sphere of radius R, with $f(u, v) = (R^2 - u^2 - v^2)^{1/2}$

Problem 20. Let V be a small neighborhood of the point p on the regular surface S, and $N(V) \subset S^2$ its image under the Gauss map N.

Show that the Gaussian curvature K of S at p is given by the limit

$$K = \lim_{V \to p} \text{area}(N(V)) / \text{area}(V) .$$

Show that this generalizes an analogous result for the curvature of plane curves.