Math 501 - Differential Geometry
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6. GEODESICS

In the Euclidean plane, a straight line can be characterized in two different ways:

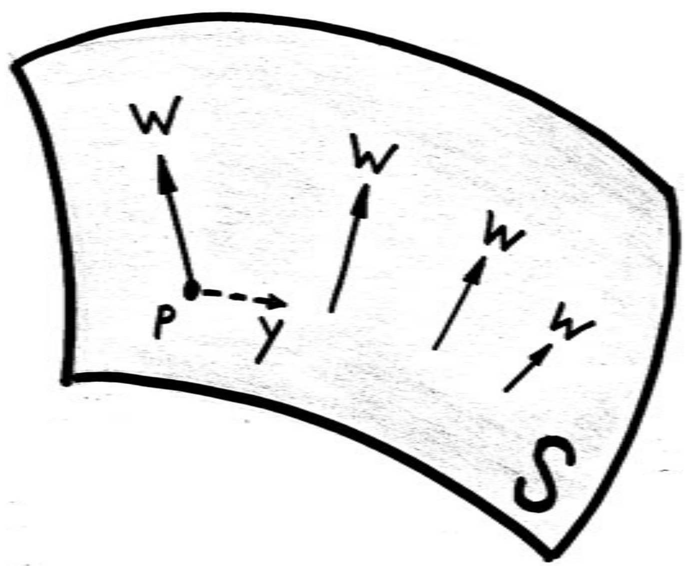
(1) it is the shortest path between any two points on it;

(2) it bends neither to the left nor the right (that is, it has zero curvature) as you travel along it.

We will transfer these ideas to a regular surface in 3-space, where ***geodesics*** play the role of straight lines.

Covariant derivatives.

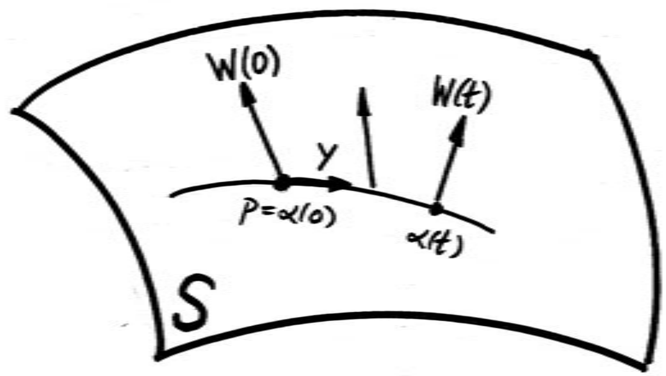
To begin, let S be a regular surface in R^3, and let W be a smooth tangent vector field defined on S.



If p is a point of S and Y is a tangent vector to S at p, that is, $Y \in T_pS$, we want to figure out how to measure the rate of change of W at p with respect to Y.

Let $\alpha(t)$ be a smooth curve on S defined for t in some neighborhood of 0, with $\alpha(0) = p$, and $\alpha'(0) = Y$.

Then $W(\alpha(t)) = W(t)$ is a vector field along the curve α.



We define

$$(DW/dt)(p) = \text{orthog proj of } dW/dt|_{t=0} \text{ onto } T_pS$$

and call this the ***covariant derivative*** of the vector field W at the point p with respect to the vector Y.

The above definition makes use of the extrinsic geometry of S by taking the ordinary derivative dW/dt in R^3, and then projecting it onto the tangent plane to S at p.

But we will see that, in spite of appearances, the covariant derivative DW/dt depends only on the intrinsic geometry of S.

To show that the covariant derivative depends only on the intrinsic geometry of S, and also that it depends only on the tangent vector Y (not the curve α), we will obtain a formula for DW/dt in terms of a parametrization $X(u,v)$ of S near p.

Let $\alpha(t) = X(u(t), v(t))$, and write

$$W(t) = a(u(t), v(t))\, X_u + b(u(t), v(t))\, X_v$$
$$= a(t)\, X_u + b(t)\, X_v .$$

Then by the chain rule,

$$dW/dt = W'(t) = a'\, X_u + a\, (X_u)' + b'\, X_v + b\, (X_v)'$$
$$= a'\, X_u + a\, (X_{uu}\, u' + X_{uv}\, v')$$
$$+ b'\, X_v + b\, (X_{vu}\, u' + X_{vv}\, v') .$$

Recall that

$$X_{uu} = \Gamma^1_{11} X_u + \Gamma^2_{11} X_v + e\,N$$
$$X_{uv} = \Gamma^1_{12} X_u + \Gamma^2_{12} X_v + f\,N$$
$$X_{vu} = \Gamma^1_{21} X_u + \Gamma^2_{21} X_v + f\,N$$
$$X_{vv} = \Gamma^1_{22} X_u + \Gamma^2_{22} X_v + g\,N\,.$$

Inserting these values into the formula for dW/dt and dropping each appearance of N, we get

$$DW/dt = (a' + a\Gamma^1_{11}u' + a\Gamma^1_{12}v' + b\Gamma^1_{21}u' + b\Gamma^1_{22}v')\,X_u$$

$$+ \; (b' + a\Gamma^2_{11}u' + a\Gamma^2_{12}v' + b\Gamma^2_{21}u' + b\Gamma^2_{22}v')\,X_v\,.$$

We repeat the formula:

$$DW/dt = (a' + a\Gamma^1_{11}u' + a\Gamma^1_{12}v' + b\Gamma^1_{21}u' + b\Gamma^1_{22}v')\, X_u$$

$$+ (b' + a\Gamma^2_{11}u' + a\Gamma^2_{12}v' + b\Gamma^2_{21}u' + b\Gamma^2_{22}v')\, X_v .$$

From this formula, we learn two things:

(1) The covariant derivative DW/dt depends only on the tangent vector $Y = X_u u' + X_v v'$ and not on the specific curve α used to "represent" it.

(2) The covariant derivative DW/dt depends only on the intrinsic geometry of the surface S, because the Christoffel symbols Γ^k_{ij} are already known to be intrinsic.

Tensor notation.

This is a good time to display the advantages of tensor notation.

Notation used above	***Tensor notation***
X_u and X_v	$X_{,1}$ and $X_{,2}$
$W = a X_u + b X_v$	$W = w^1 X_{,1} + w^2 X_{,2}$
	$= w^i X_{,i}$
$Y = u' X_u + v' X_v$	$Y = y^i X_{,i}$
a' and b'	$Y(w^1)$ and $Y(w^2)$
DW/dt	$D_Y W$ (or $\nabla_Y W$)

Formula for covariant derivative

$$DW/dt = (a' + a\Gamma^1{}_{11}u' + a\Gamma^1{}_{12}v' + b\Gamma^1{}_{21}u' + b\Gamma^1{}_{22}v')\, X_u$$

$$+ (b' + a\Gamma^2{}_{11}u' + a\Gamma^2{}_{12}v' + b\Gamma^2{}_{21}u' + b\Gamma^2{}_{22}v')\, X_v\ .$$

Same formula in tensor notation

$$D_Y W = (Y(w^k) + w^i\, \Gamma^k{}_{ij}\, y^j)\, X_{,k}\ .$$

Parallel vector fields and parallel transport.

Let S be a regular surface in R^3, and $\alpha: I \to S$ a smooth curve in S. A vector field W ***along*** α is a choice of tangent vector $W(t) \in T_{\alpha(t)}S$ for each $t \in I$.

This vector field is ***smooth*** if we can write

$$W(t) = a(t) X_u + b(t) X_v$$

in local coordinates, with $a(t)$ and $b(t)$ smooth fns of t.

Problem 1. Check that this definition of smoothness of a vector field along α is independent of the choice of local coordinates for S.

Example. The velocity vector field $\alpha'(t)$ is an example of a smooth vector field along α .

If W is a smooth vector field along the smooth curve α on S , then the expression

$$DW/dt = (a' + a\Gamma^1_{11}u' + a\Gamma^1_{12}v' + b\Gamma^1_{21}u' + b\Gamma^1_{22}v')\, X_u$$

$$+ (b' + a\Gamma^2_{11}u' + a\Gamma^2_{12}v' + b\Gamma^2_{21}u' + b\Gamma^2_{22}v')\, X_v$$

is well-defined and is called the ***covariant derivative*** of W along α . As before, DW/dt is simply the orthogonal projection of dW/dt onto T_pS .

Example. Let α be a smooth curve on the regular surface S, with velocity vector field $\alpha'(t)$. The covariant derivative $D\alpha'/dt$ is the portion of the acceleration $d\alpha'/dt = \alpha''(t)$ which is tangent to S.

Definition. A smooth vector field W defined along a smooth curve $\alpha: I \to S$ is said to be ***parallel*** if

$$DW/dt = 0 \text{ for all } t \in I .$$

Problem 2. Show that a vector field W defined along a curve α in the plane R^2 is parallel along α if and only if W is constant.

Problem 3. Let V and W be parallel vector fields along a curve $\alpha: I \rightarrow S$. Show that the inner product $< V , W >$ is constant along α . Conclude that the lengths $|V|$ and $|W|$ are also constant along α .

Problem 4. Let $\alpha: I \rightarrow S^2$ parametrize a great circle at constant speed. Show that the velocity field α' is parallel along α .

Proposition. ***Let*** $\alpha: I \rightarrow S$ ***be a smooth curve on the regular surface*** S ***.*** ***Let*** W_0 ***be an arbitrary tangent vector to*** S ***at*** $\alpha(t_0)$ ***.*** ***Then there is a unique parallel vector field*** $W(t)$ ***along*** α ***with*** $W(t_0) = W_0$ ***.***

Proof. Working in local coordinates $X: U \rightarrow S$, we can write $\alpha(t) = X(u(t), v(t))$. Let

$$W(t) = a(t)\, X_u + b(t)\, X_v$$

be the vector field we seek.

Then, since

$$DW/dt = (a' + a\Gamma^1_{11}u' + a\Gamma^1_{12}v' + b\Gamma^1_{21}u' + b\Gamma^1_{22}v')\, X_u + (b' + a\Gamma^2_{11}u' + a\Gamma^2_{12}v' + b\Gamma^2_{21}u' + b\Gamma^2_{22}v')\, X_v ,$$

the condition that $W(t)$ be parallel along α is that

$$a' + a\Gamma^1_{11}u' + a\Gamma^1_{12}v' + b\Gamma^1_{21}u' + b\Gamma^1_{22}v' = 0$$
$$b' + a\Gamma^2_{11}u' + a\Gamma^2_{12}v' + b\Gamma^2_{21}u' + b\Gamma^2_{22}v' = 0 .$$

This is a system of two first order linear ODEs for the unknown functions $a(t)$ and $b(t)$. By standard theorems, a solution exists and is unique, with given initial condition $W_0 = a(t_0)\, X_u + b(t_0)\, X_v$.

Remark. This proposition allows us to talk about ***parallel transport*** of a given tangent vector $W_0 \in T_pS$ along a curve α on S which passes through p .

Problem 5. Let α be a smooth curve on S connecting the points p and q . Show that parallel transport along α is an isometry from T_pS to T_qS .

Problem 6. Show that if two surfaces are tangent along a common curve α , then parallel transport along α is the same for both surfaces.

Problem. Explain how to carry out parallel transport along ***piecewise smooth curves***.

Geodesics.

Definition. Let S be a regular surface in R^3. A smooth curve $\gamma: I \to S$ is called a ***geodesic*** if the field of its tangent vectors $\gamma'(t)$ is parallel along γ, that is, if

$$D\gamma'/dt = 0 .$$

Note that we can also write this equation as

$$D_{\gamma'}\gamma' = 0 \quad \text{or} \quad \nabla_{\gamma'}\gamma' = 0 .$$

Remarks.

- The geodesics on the plane R^2 are just the straight lines, travelled at constant speed.

- Every geodesic on a surface is travelled at constant speed.

- A straight line which lies on a surface is automatically a geodesic.

- A smooth curve on a surface is a geodesic if and only if its acceleration vector is normal to the surface.

- The geodesics on a round sphere are the great circles.

Problem 7. (a) Find as many geodesics as you can on the right circular cylinder $x^2 + y^2 = 1$ in R^3.

(b) Observe that there can be infinitely many geodesics connecting two given points on this cylinder.

Next we want to define the ***geodesic curvature*** of a curve on a regular surface. Before doing that, let's recall how we defined curvature of curves in R^3 and R^2.

If $\alpha: I \to R^3$ is a smooth curve parametrized by arc length, we defined the ***curvature*** of α at s to be the real number $\kappa(s) = |\alpha''(s)|$. There is no way to give a sign to the curvature of a curve in R^3.

But if $\alpha: I \to R^2$ is a smooth plane curve parametrized by arc length, we can give a sign to its curvature as follows.

Orient the plane R^2 . Let $T(s) = \alpha'(s)$ be the unit tangent vector to the curve at $\alpha(s)$. Let $N(s)$ to be the unit vector normal to $T(s)$ such that the ordered O.N. basis $T(s)$, $N(s)$ agrees with the chosen orientation of R^2 .

Then define $\kappa(s) = < \alpha''(s) , N(s) >$.

With the usual orientation of R^2 , positive curvature indicates the curve is bending to the left as you go ahead; negative indicates bending to the right.

We can do the same thing on an oriented regular surface S in R^3 , as follows.

Let α: $I \to S$ be a smooth curve on S , parametrized by arc length. Let $T(s) = \alpha'(s)$ be the unit tangent vector to the curve at $\alpha(s)$. Let $M(s)$ be the unit vector at $\alpha(s)$ which is tangent to the surface S but orthogonal to $T(s)$, chosen so that the ordered O.N. basis $T(s)$, $M(s)$ agrees with the chosen orientation of $T_{\alpha(s)} S$.

If we have already chosen a unit surface normal N for our surface S , then we can simply let $M(s) = N(\alpha(s)) \times T(s)$. That way, the ordered O.N. basis $T(s)$, $M(s)$, $N(\alpha(s))$ agrees with the orientation of R^3 .

Now we define the ***geodesic curvature*** of the curve α at the point $\alpha(s)$ to be

$$\kappa_g(s) = < \alpha''(s) , M(s) > .$$

Note that

$$< \alpha''(s) , M(s) > = < d\alpha'/ds , M(s) > = < D\alpha'/ds , M(s) > .$$

Thus a smooth curve $\gamma: I \to S$ parametrized by arc length is a geodesic if and only if its geodesic curvature is zero.

Problem 8. Show that the geodesic curvature of the curve $\alpha: I \to S$ at the point $\alpha(s)$ is the same as the ordinary curvature at that point of the plane curve obtained by projecting α orthogonally onto the tangent plane $T_{\alpha(s)}S$.

Solution. Shift parameters so that $s = 0$ at the point in question, assume that $\alpha(0)$ is at the origin of our coordinate system, and that the unit tangent vector T to this curve, the normal M within the surface, and the surface normal N line up with the x , y and z axes.

Assume that s is an arc length parameter along our curve.

Write $\alpha(s) = (x(s), y(s), z(s))$.

Then $\alpha'(0) = (x'(0), y'(0), z'(0)) = (1, 0, 0)$

and $\alpha''(0) = (x''(0), y''(0), z''(0)) = (0, b, c)$.

When $s = 0$, the curvature of this curve in 3-space is

$$|\alpha''(0)| = (b^2 + c^2)^{1/2}$$

while its geodesic curvature on the surface is b .

Projecting our curve onto the tangent plane to the surface at the given point yields the curve

$$\beta(s) = (x(s), y(s), 0) ,$$

where s is no longer an arc length parameter.

The unsigned curvature of β at the given point is

$$\kappa(0) = |\beta'(0) \times \beta''(0)| / |\beta'(0)|^3$$

$$= |(1, 0, 0) \times (0, b, 0)| / 1^3 = |b| .$$

Its signed curvature there is b, the same as the geodesic curvature of α on our surface, completing the argument.

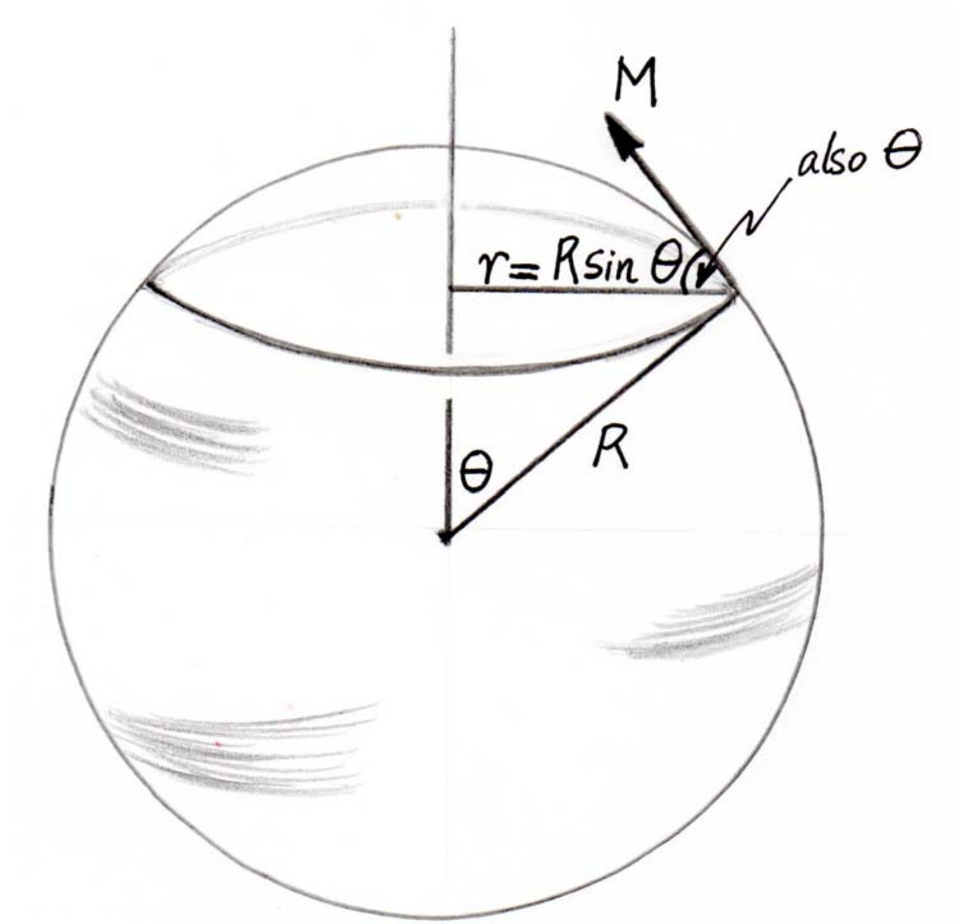
Problem 9. Let $\alpha: I \rightarrow S$ be a smooth curve on the regular surface S in R^3 .

Let $\kappa(s)$ be the ordinary curvature of the curve α in R^3 , let $\kappa_g(s)$ be its geodesic curvature on the surface S , and let $k_n(s)$ be the normal curvature of the surface S at the point $\alpha(s)$ in the direction $\alpha'(s)$. Show that

$$\kappa(s)^2 = \kappa_g(s)^2 + k_n(s)^2 .$$

Check this when α is a small circle on a round sphere.

Solution. Let's do the example first on a sphere of radius R , as shown below.



The small circle shown has radius $r = R \sin\theta$,
and when parametrized by arc length is given by

$$\alpha(s) = (r \cos(s/r) , r \sin(s/r) , R \cos\theta) .$$

Then $\alpha'(s) = (-\sin(s/r) , \cos(s/r) , 0)$ and

$$\alpha''(s) = (-(1/r) \cos(s/r) , -(1/r) \sin(s/r) , 0) .$$

The curvature of this small circle is $\kappa = 1/r = 1 / (R \sin\theta)$.

Its geodesic curvature is

$$\kappa_g = \alpha'' \bullet M = (1/r) \cos\theta = \cos\theta / (R \sin\theta) .$$

The normal curvature of our surface is $\kappa_n = 1/R$.

To check that $\kappa^2 = \kappa_g^2 + \kappa_n^2$, we write

$$\kappa_g^2 + \kappa_n^2 = (\cos^2\theta / (R^2 \sin^2\theta)) + 1/R^2$$

$$= (\cos^2\theta + \sin^2\theta) / (R^2 \sin^2\theta)$$

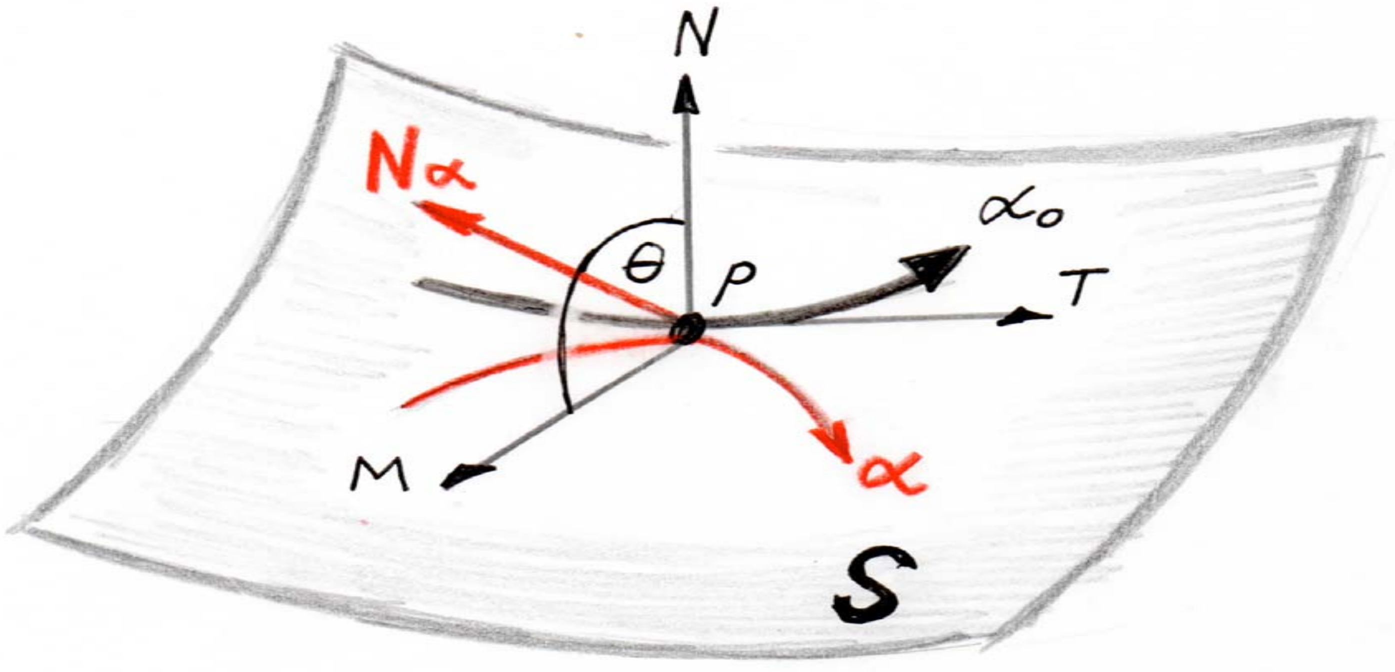
$$= 1 / (R^2 \sin^2\theta) = \kappa^2 ,$$

as desired.

The proof of the formula

$$\kappa(s)^2 = \kappa_g(s)^2 + k_n(s)^2$$

is an application of Meusnier's Theorem.



The curve α shown above passes through the point p on the surface S with tangent vector T and principal normal N_α . Its curvature there is κ .

The orthormal frame T, M, N consists of the tangent vector T , an orthogonal vector M still tangent to S , and the surface normal N .

The plane spanned by T and N cuts the surface along the curve α_o , whose curvature k_n is the normal curvature of the surface S at p .

By Meusnier's Theorem, the curvature κ of the curve α at p is related to the normal curvature k_n by the formula

$$\kappa = k_n / \cos\theta ,$$

where θ is the angle between N_α and N, as shown in the figure above.

Assuming α is parametrized by arc length, its geodesic curvature κ_g at p is by definition

$$\kappa_g = < \alpha'' , M > = < \kappa N_\alpha , M > = \kappa \cos(\pi/2 - \theta) = \kappa \sin\theta .$$

Then $$\kappa_g^2 + k_n^2 = \kappa^2 \sin^2\theta + \kappa^2 \cos^2\theta = \kappa^2 ,$$

as desired.

THEOREM (Existence and uniqueness of geodesics).
Let* $\mathbf{S}$ *be a regular surface in* $\mathbf{R}^3$ *,* $\mathbf{p}$ *a point on* $\mathbf{S}$ *, and* $\mathbf{W} \neq \mathbf{0}$ *a tangent vector to* $\mathbf{S}$ *at* $\mathbf{p}$ *. Then there is an* $\boldsymbol{\varepsilon} > \mathbf{0}$ *and a unique geodesic* $\boldsymbol{\gamma}: (-\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}) \to \mathbf{S}$ *such that

$$\boldsymbol{\gamma}(0) = \mathbf{p} \qquad \textit{and} \qquad \boldsymbol{\gamma}'(0) = \mathbf{W} .$$

Proof. Using local coordinates $X: U \to S$, let us write the geodesic to be found as $\gamma(t) = X(u(t), v(t))$. Then $\gamma'(t) = X_u u'(t) + X_v v'(t)$.

In our now familiar formula for the covariant derivative,

$$DW/dt = (a' + a\Gamma^1_{11}u' + a\Gamma^1_{12}v' + b\Gamma^1_{21}u' + b\Gamma^1_{22}v')\, X_u + (b' + a\Gamma^2_{11}u' + a\Gamma^2_{12}v' + b\Gamma^2_{21}u' + b\Gamma^2_{22}v')\, X_v\ ,$$

the role of the vector $W = a\, X_u + b\, X_v$ will be played by $\gamma'(t) = X_u\, u'(t) + X_v\, v'(t)$, and the role of the vector $Y = u'\, X_u + v'\, X_v$ will also be played by $\gamma'(t)$.

In other words,

$$a = u'\ , \quad b = v'\ , \quad a' = u'' \quad \text{and} \quad b' = v''\ .$$

Thus

$$DW/dt = D\gamma'/dt = D_{\gamma'}\gamma'$$

$$= \quad (u'' + u'\,\Gamma^1{}_{11}u' + u'\,\Gamma^1{}_{12}v' + v'\,\Gamma^1{}_{21}u' + v'\,\Gamma^1{}_{22}v')\,X_u$$

$$+ \; (v'' + u'\,\Gamma^2{}_{11}u' + u'\,\Gamma^2{}_{12}v' + v'\,\Gamma^2{}_{21}u' + v'\,\Gamma^2{}_{22}v')\,X_v\,.$$

So the system of ODEs to be satisfied by a geodesic is

$$u'' + u'\,\Gamma^1{}_{11}u' + u'\,\Gamma^1{}_{12}v' + v'\,\Gamma^1{}_{21}u' + v'\,\Gamma^1{}_{22}v' = 0\,,$$

$$v'' + u'\,\Gamma^2{}_{11}u' + u'\,\Gamma^2{}_{12}v' + v'\,\Gamma^2{}_{21}u' + v'\,\Gamma^2{}_{22}v' = 0\,.$$

The standard existence and uniqueness theorem for such systems of ODEs promises us a unique solution $u = u(t)$, $v = v(t)$ defined on some interval $(-\varepsilon, \varepsilon)$ and satisfying the initial conditions

$$u(0) = u_0, \; v(0) = v_0, \; u'(0) = u_0' \text{ and } v'(0) = v_0',$$

$$\text{where } X(u_0, v_0) = p \text{ and } dX(u_0', v_0') = W.$$

This completes the proof.

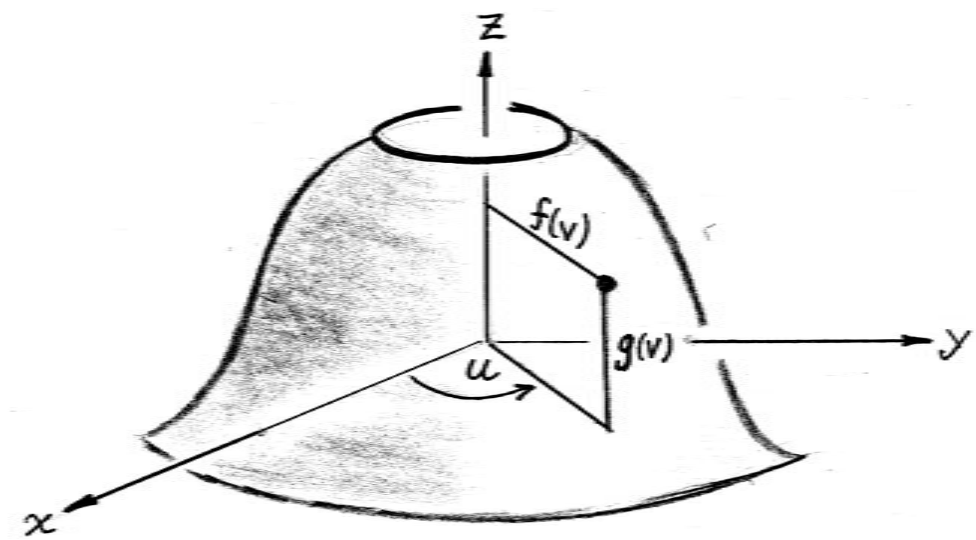
Problem 10. Show that in tensor notation, the two equations for a geodesic $\gamma(t) = X(u^1(t), u^2(t))$ are

$$(u^k)'' + (u^i)' \Gamma^k_{ij} (u^j)' = 0, \qquad \text{for } k = 1, 2.$$

Example - Geodesics on a surface of revolution.

Consider the surface of revolution parametrized by

$$X(u, v) = (f(v) \cos u ,\ f(v) \sin u ,\ g(v)) .$$



$$X_u = (- f \sin u ,\ f \cos u ,\ 0)$$
$$X_v = (f\,' \cos u ,\ f\,' \sin u ,\ g')$$

$$E = \langle X_u , X_u \rangle = f^2$$

$$F = \langle X_u , X_v \rangle = 0$$

$$G = \langle X_v , X_v \rangle = f'^2 + g'^2$$

$$\Gamma^1_{11} = 0 \qquad \Gamma^2_{11} = -f f' / (f'^2 + g'^2)$$

$$\Gamma^1_{12} = f' / f \qquad \Gamma^2_{12} = 0$$

$$\Gamma^1_{22} = 0 \qquad \Gamma^2_{22} = (f' f'' + g' g'') / (f'^2 + g'^2)$$

The geodesic equations are

$$(1)\quad u'' + u'\,\Gamma^1{}_{11}\,u' + u'\,\Gamma^1{}_{12}\,v' + v'\,\Gamma^1{}_{21}\,u' + v'\,\Gamma^1{}_{22}\,v' = 0$$

$$(2)\quad v'' + u'\,\Gamma^2{}_{11}\,u' + u'\,\Gamma^2{}_{12}\,v' + v'\,\Gamma^2{}_{21}\,u' + v'\,\Gamma^2{}_{22}\,v' = 0 .$$

Inserting the actual values for the Christoffel symbols gives

$$(1)\quad u'' + 2\,(f\,'/f)\,u'\,v' = 0$$

$$(2)\quad v'' + (-f\,f\,'\,/\,(f\,'^2 + g'^2))\,u'^2$$

$$+ ((f\,'\,f\,'' + g'\,g'')\,/\,(f\,'^2 + g'^2))\,v'^2 = 0 .$$

Caution about the notation:

$f\,' = df/dv \quad f\,'' = d^2f/dv^2$ and likewise for g, but
$u' = du/dt \quad u'' = d^2u/dt^2$ and likewise for v.

Problem 11. Check that $u = \text{constant}$ and $v = v(t)$ is a solution of the geodesic equations for some choice of $v(t)$.

Hint. Equation (1) above is automatically satisfied, and equation (2) simply determines $v(t)$ so that the curve is travelled at constant speed.

Problem 12. Show that the curve $X(u(t), v(t))$ on our surface of revolution is travelled at constant speed if and only if

$$(3) \qquad u'' f^2 u' + v'' (f'^2 + g'^2) v' + (f' f'' + g' g'') v'^3 + f f' u'^2 v' = 0 .$$

Problem 13. Show that equations (1) and (2) together imply equation (3) .

Problem 14. Show that if $f'(v_0) = 0$, then the circle $u = ct$ and $v = v_0$ satisfies equations (1) and (2) , and is hence a geodesic.

Problem 15. Show that if $v' \neq 0$, then equations (1) and (3) together imply equation (2) . So to get a geodesic, just satisfy equation (1) and make sure you travel at constant speed.

The issue now is to interpret equation

$$(1)\quad u'' + 2\,(f\,'/f)\,u'\,v' = 0\,.$$

This equation implies that

$$\begin{aligned}(f^2\,u')' &= f^2\,u'' + 2\,f\,f\,'\,v'\,u' \\ &= f^2\,(u'' + 2\,(f\,'/f)\,u'\,v') = 0\,,\end{aligned}$$

which tells us that $\qquad f^2\,u' = \text{constant}\,.$

To see the meaning of this, imagine that we travel at constant speed c along the curve $X(u(t),\,v(t))$ on our surface of revolution.

Let $\alpha(t)$ denote the angle that our curve makes with the horizontal circle on the surface through the given point.

Then on the one hand,

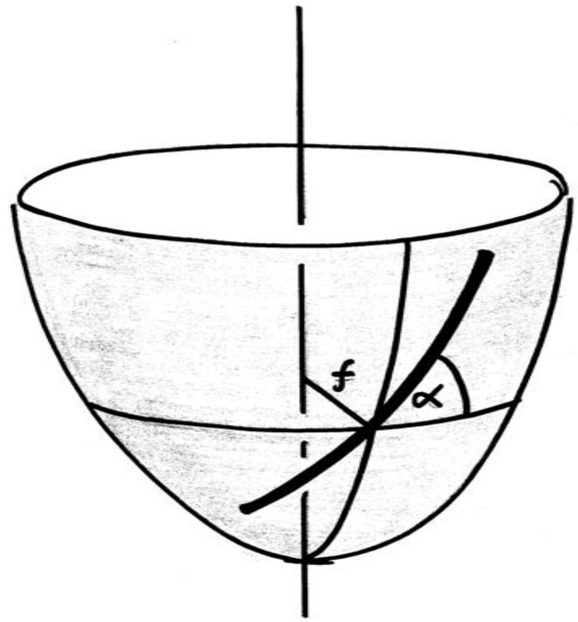
$$< X_u u' + X_v v' , X_u > \; = \; < X_u , X_u > u' \; = \; f^2 u'$$

while on the other hand, this inner product equals

$$| X_u u' + X_v v'| \; |X_u| \cos \alpha \; = \; c \; f \; \cos \alpha \; .$$

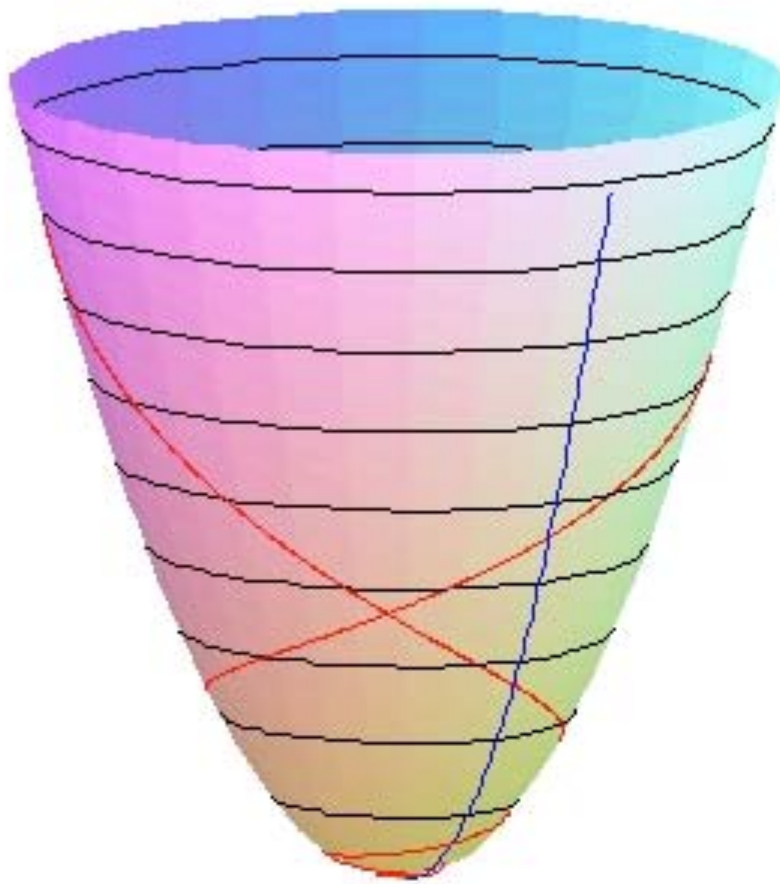
So the equation $f^2 u' = \text{constant}$ is equivalent to

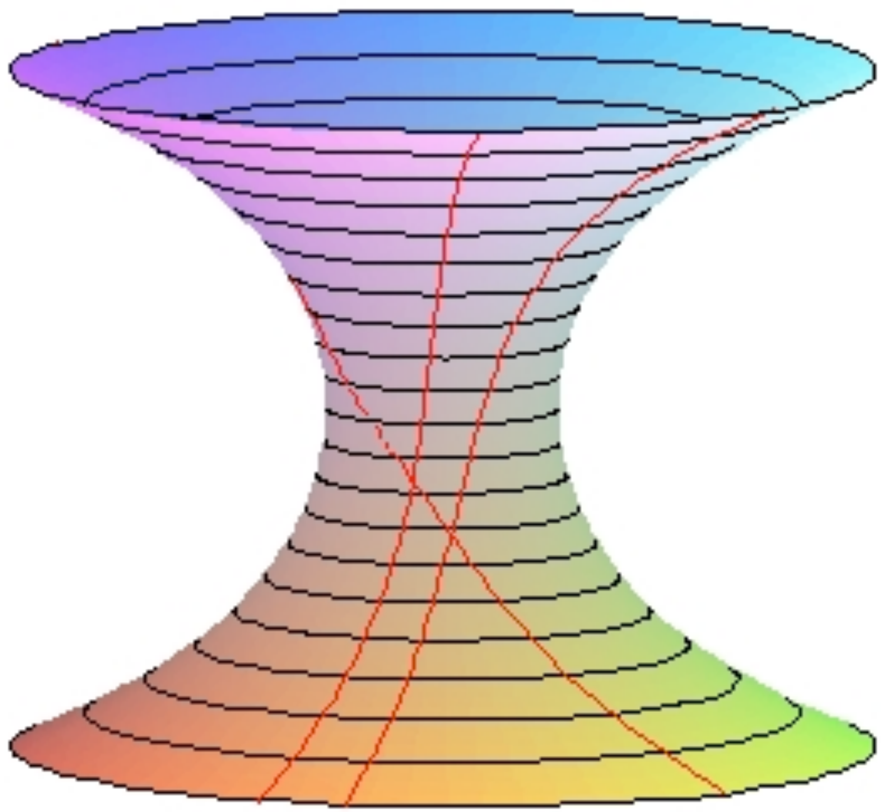
(4) $$f \cos \alpha \; = \; \text{constant} \; .$$



CLAIRAUT'S THEOREM. *Geodesics on the surface of revolution* $X(u, v) = (f(v) \cos u , f(v) \sin u , g(v))$ *are characterized by the equation*

$$f \cos \alpha = \text{constant} .$$





Comments.

- The value of $f \cos \alpha$ is constant along a given geodesic, but different geodesics may have different constants.

- If we consider all geodesics through a given point

$$X(u_0, v_0) = (f(v_0) \cos u_0 , \; f(v_0) \sin u_0 , \; g(v_0))$$

on the surface, then

$$- f(v_0) \le \text{constant} \le f(v_0) .$$

The extreme constants $-f(v_0)$ and $f(v_0)$ correspond to geodesics through $X(u_0, v_0)$ which at that point are tangent to the horizontal circle.

The constant 0 corresponds to the vertical geodesic through $X(u_0, v_0)$, which is simply the profile curve $u = u_0$.

- Traveling along a given geodesic, as the surface moves farther away from the z-axis, the geodesic becomes more vertical.