Hour Exam - II.

Part I.

1. C.

The only singularity of $\frac{1}{1-z}$ is at $z=1$

Therefore the radius of convergence

$= |i-1| = \sqrt{2}$

2. A

Expanding in terms of $z-2$:

$$\frac{e^z}{z-2} = \frac{e^{z-2+2}}{z-2} = e^2 \cdot \frac{e^{z-2}}{z-2} = e^2 \cdot \frac{\sum_{n=0}^{\infty} \frac{1}{n!}(z-2)^n}{z-2}$$

$$= \frac{e^2}{z-2} + \sum_{n=0}^{\infty} \frac{e^2}{(n+1)!}(z-2)^n$$

3. B.

For $1-e^z$, $1-e^0=0$, $(1-e^z)' = -e^z\Big|_{z=0} = -1 \neq 0$

$\Rightarrow 0$ is a zero of order 1

For z^4, 0 is a zero of order 4

$\Rightarrow$ 0 is a pole of order $4-1=3$ for $\frac{1-e^z}{z^4}$.

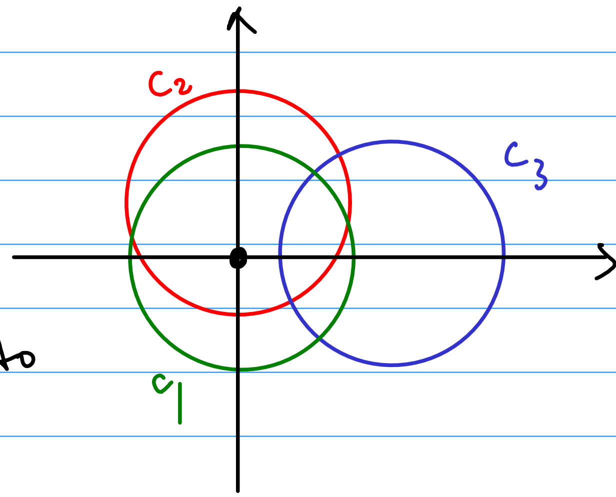
4. A.

Notice e^{-1/z^2} has an essential singularity at $z=0$.

$z^5 e^{-1/z^2}$ is analytic away from 0.

In the complex plane, c_1 & c_2 can be

deformed to each other
without touching the
singularity 0.



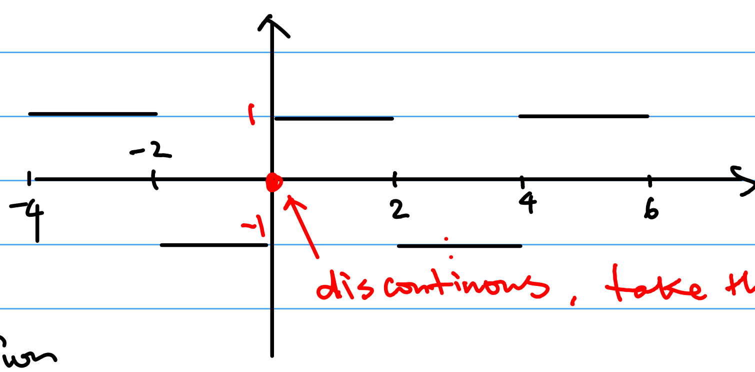
But they will touch 0
if they want to deform to
C_3

$$\Rightarrow \quad \oint_{C_1} = \oint_{C_2} \neq \oint_{C_3}.$$

5. A Did it in class.

6. C.
sinx has periodicity of 2π, therefore can't be obtained from Fourier series of periodicity of π

7. C



After extension
we have

$$f(x+4) = f(x) \Rightarrow f(15) = f(-1) = -1, \; F(15) = -1$$

$$F(8) = F(24) = F(0) = \frac{1+(-1)}{2} = 0.$$

8. B

9. (a). $\frac{7z-3}{z(z-1)} = \frac{1}{z} \quad \frac{7z-3}{z-1}$

$= \frac{1}{z}\left(7 - \frac{4}{1-z}\right)$

$= \frac{1}{z}\left(7 - 4\sum_{n=0}^{\infty} z^n\right)$

$= \frac{3}{z} - 4\sum_{n=0}^{\infty} z^n$

(b). $\frac{7z-3}{z(z-1)} = \frac{1}{z-1}\left(7 - \frac{3}{z}\right)$

$= \frac{1}{z-1}\left(7 - \frac{3}{1+(z-1)}\right)$

$= \frac{1}{z-1}\left[7 - 3\cdot\sum_{n=0}^{\infty}(-1)^n(z-1)^n\right]$

$= \frac{4}{z-1} + 3\sum_{n=0}^{\infty}(-1)^n(z-1)^n$

10. $T=2 \qquad f(x) = \sum_{n=-\infty}^{\infty} C_n e^{in\pi x}$

$C_n = \frac{1}{2}\int_{-1}^{1} f(x)\, e^{-in\pi x}$

$= \frac{1}{2}\left[\int_{-1}^{0} e^{x(1-in\pi)}dx + \int_{0}^{1} e^{-x(1+in\pi)}dx\right.$

$= \frac{1}{2}\left[\frac{1-e^{-1+in\pi}}{1-in\pi} + \frac{e^{-1-in\pi}-1}{-(1+in\pi)}\right]$

$= \frac{1}{2}\left[1-(-1)^n e^{-1}\right]\cdot\left(\frac{1}{1-in\pi} + \frac{1}{1+in\pi}\right)$

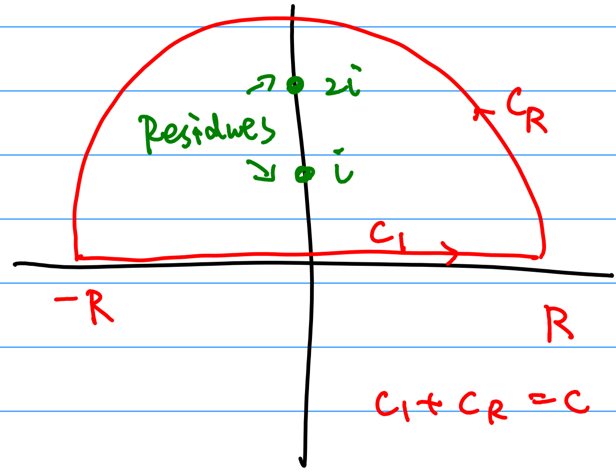
$= \frac{1-(-1)^n e^{-1}}{1+n^2\pi^2}$

11. $$\int_{-\infty}^{+\infty} \frac{\cos x}{(x^2+1)(x^2+4)}\,dx$$

$$= \text{P.V.} \int_{-\infty}^{+\infty} \frac{\cos x}{(x^2+1)(x^2+4)}\,dx$$

$$= \lim_{R\to\infty} \int_{-R}^{R} \frac{\cos x}{(x^2+1)(x^4+4)}\,dx$$

$$= \text{Re} \lim_{R\to\infty} \int_{-R}^{R} \frac{e^{ix}}{(x^2+1)(x^2+4)}\,dx$$



Let $$f(z) = \frac{e^{iz}}{(z^2+1)(z^2+4)}$$

$$\oint_C f(z)\,dz = \int_{-R}^{R} f(z)\,dz + \int_{C_R} f(z)\,dz$$

$$\Rightarrow \lim_{R\to\infty} \int_{-R}^{R} f(z)\,dz = \lim_{R\to\infty} \oint_C f(z)\,dz - \lim_{R\to\infty} \int_{C_R} f(z)\,dz$$

Since $f(z) = \dfrac{e^{iz}}{(z^2+1)(z^2+4)}$. according to Thm 19.16

$$\lim_{R\to\infty} \int_{C_R} f(z)\,dz = 0$$

$$\Rightarrow \lim_{R\to\infty} \int_{-R}^{R} f(z)\,dz = \lim_{R\to\infty} \oint_C f(z)\,dz$$

$$= 2\pi i \left[\text{Res}(f, i) + \text{Res}(f, 2i)\right]$$

$$= 2\pi i \left[\frac{e^{i\cdot i}}{2i\cdot(-1+4)} + \frac{e^{i(2i)}}{(-4+1)\cdot 4i}\right]$$

$$= \pi\left(\frac{e^{-1}}{3} - \frac{e^{-2}}{6}\right)$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{\cos x}{(x^2+1)(x^2+4)}\,dx = \pi\left(\frac{e^{-1}}{3} - \frac{e^{-2}}{6}\right)$$

12. $T = 1, \quad P = 1/2$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(2n\pi x) + b_n \sin(2n\pi x)$$

$$a_0 = 2\int_0^1 f(x)\,dx = 2\int_0^1 (x+1)\,dx = 3$$

$$a_n = 2\int_0^1 f(x)(\cos 2n\pi x)\,dx$$

$$= 2\int_0^1 x\cos(2n\pi x)\,dx + 2\int_0^1 \cos(2n\pi x)\,dx$$

$$= \frac{1}{n\pi} x\sin 2n\pi x\Big|_0^1 - \frac{1}{n\pi}\int_0^1 \sin 2n\pi x\,dx + 0$$

$$= 0 - 0 + 0 = 0$$

$$b_n = 2\int_0^1 x\cdot\sin(2n\pi x)\,dx + 2\int_0^1 \sin(2n\pi x)\,dx$$

$$= -\frac{1}{n\pi} x\cos 2n\pi x\Big|_0^1 + \frac{1}{n\pi}\int_0^1 \cos 2n\pi x\,dx + 0$$

$$= -\frac{1}{n\pi} + 0 + 0 = -\frac{1}{n\pi}.$$

$$\Rightarrow f(x) = \frac{3}{2} - \sum_{n=1}^{\infty} \frac{1}{n\pi}\sin(2n\pi x)$$