13.6.4

We have equation

$$k\frac{\partial^2 u}{\partial x^2} + r = \frac{\partial u}{\partial t}$$

subject to the boundary condition

$$u(0,t) = u_0, \quad u(1,t) = u_1, \quad u(x,0) = f(x).$$

This is a time independent nonhomogeneous BVP problem. Following the example 1 in the book, we write $u(x,t) = v(x,t) + \psi(x)$ with $v(x,t)$ satisfying the homogeneous BVP and $k\psi''(x) + r = 0$ with $\psi(0) = u_0$ and $\psi_1 = u_1$. It's easy to solve

$$\psi(x) = u_0(1-x) - \frac{r}{2k}(x^2 - x) + u_1 x = -\frac{r}{2k}x^2 + (u_1 + \frac{r}{2k} - u_0)x + u_0.$$

Using this, we have

$$v(x,0) = u(x,0) - \psi(x) = f(x) + \frac{r}{2k}x^2 - (u_1 + \frac{r}{2k} - u_0)x - u_0.$$

Solving v in the usual manner we have

$$v(x,t) = \sum_{n=1}^{\infty} A_n \sin(n\pi x) e^{-kn^2\pi^2 t},$$

where

$$A_n = 2\int_0^1 \left[f(x) + \frac{r}{2k}x^2 - (u_1 + \frac{r}{2k} - u_0)x - u_0\right] \sin(n\pi x)\,\mathrm{d}x.$$

And in terms of these A_n we have

$$u(x,t) = -\frac{r}{2k}x^2 + (u_1 + \frac{r}{2k} - u_0)x + u_0 + \sum_{n=1}^{\infty} A_n \sin(n\pi x) e^{-kn^2\pi^2 t}.$$

13.6.12

We have the equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = -h, \quad h > 0$$

on $(0,\pi) \times (0,\infty)$ subject to the boundary condition

$$u(0,y) = 0, \quad u(\pi,y) = 1, \quad u(x,0) = 0.$$

Well, the inhomogeneous part is y independent, therefore we can write

$$u(x,y) = v(x,y) + \psi(x)$$

with $v(x,y)$ satisfying the homogeneous BVP and

$$\psi''(x) = -h$$

with $\psi(0) = 0$ and $\psi(\pi) = 1$. Solving ψ we get

$$\psi(x) = -\frac{h}{2}(x^2 - x) + x = -\frac{h}{2}x^2 + (\frac{h}{2} + 1)x.$$

13.6.12 continued on next page...

Using this we have

$$v(x,0) = u(x,0) - \psi(x) = \frac{h}{2}x^2 - (\frac{h}{2}+1)x.$$

v also has a secret boundary condition that $v(x,y) \to 0$ when $y \to \infty$. Solving v in the usual manner we have

$$v(x,y) = \sum_{n=1}^{\infty} A_n \sin(nx)e^{-ny},$$

where

$$A_n = \frac{2}{\pi}\int_0^{\pi} \left[\frac{h}{2}x^2 - (\frac{h}{2}+1)x\right]\sin(nx)\,\mathrm{d}x = \frac{2}{n\pi}\int_0^{n\pi}\left[\frac{h}{2n^2}x^2 - (\frac{h}{2n}+\frac{1}{n})x\right]\sin(x)\,\mathrm{d}x,$$

which we can evaluate as

$$\frac{2}{n\pi}\left\{\left[\frac{h}{2n^2}x^2 - (\frac{h}{2n}+\frac{1}{n})x\right](-\cos x) + \left[\frac{h}{n^2}x - (\frac{h}{2n}+\frac{1}{n})\right]\sin x + \frac{h}{n^2}\cos x\right\}\Bigg|_0^{n\pi} = \frac{2}{n\pi}\left\{\left[\frac{h}{n^2} - \frac{h\pi^2}{2} + (\frac{h}{2}+1)\pi\right](-1)^n - \frac{h}{n^2}\right\}.$$

Using these coefficient we have

$$u(x,y) = v(x,y) + \psi(x) = -\frac{h}{2}(x^2 - x) + x = -\frac{h}{2}x^2 + (\frac{h}{2}+1)x + \sum_{n=1}^{\infty} A_n \sin(nx)e^{-ny}.$$

13.6.18

We have the equation

$$\frac{\partial^2 u}{\partial x^2} + 2t + 3tx = \frac{\partial u}{\partial t}$$

on $(0,1)\times(0,\infty)$ subject to the boundary conditions

$$u(0,t) = t^2, \quad u(1,t) = 1, \quad u(x,0) = x^2.$$

Now the inhomogeneous part is t-dependent, therefore we can follow the example 2 and write

$$u(x,y) = v(x,y) + \psi(x,t)$$

with $\psi(x,t)$ defined as

$$\psi(x,t) = t^2 + x(1-t^2).$$

The equation for v becomes

$$\frac{\partial^2 v}{\partial x^2} + 2t + 3tx - \frac{\partial \psi}{\partial t} = \frac{\partial^2 v}{\partial x^2} + 5tx = \frac{\partial v}{\partial t}$$

with the boundary conditions

$$v(0,t) = 0, \quad v(1,t) = 0, \quad v(x,0) = x^2 - x.$$

We first expand

$$3tx = \sum_{n=1}^{\infty} B_n \sin(n\pi x).$$

$$B_n = 2\int_0^1 3tx\sin(n\pi x)\,\mathrm{d}x = \frac{6t}{n^2\pi^2}\int_0^{n\pi} x\sin x\,\mathrm{d}x = \frac{6t}{n^2\pi^2}(-x\cos x + \sin x)\Bigg|_0^{n\pi} = \frac{6(-1)^{n+1}t}{n\pi}.$$

Therefore if we expand

$$v(x,t) = \sum_{n=1}^{\infty} T_n(t)\sin(n\pi x),$$

we have for each n

$$-(n\pi)^2 T_n(t) + \frac{6(-1)^{n+1}t}{n\pi} = T_n'(t).$$

To solve this, we first rewrite it as

$$\left(e^{n^2\pi^2 t} T_n(t)\right)' = e^{n^2\pi^2 t}\frac{6(-1)^{n+1}t}{n\pi} = 6(-1)^{n+1}\left(e^{n^2\pi^2 t}\frac{t}{n^3\pi^3} - e^{n^2\pi^2 t}\frac{1}{n^5\pi^5}\right)'.$$

This means

$$T_n(t) = A_n e^{-n^2\pi^2 t} + \frac{6(-1)^{n+1}(n^2\pi^2 t - 1)}{n^5\pi^5}.$$

To get A_n, we let $t = 0$, then $T_n(0) = A_n + \frac{6(-1)^n}{n^5\pi^5}$. Hence

$$A_n = 2\int_0^1 (x^2 - x)\sin(n\pi x)\,\mathrm{d}x - \frac{6(-1)^n}{n^5\pi^5} = \frac{4[(-1)^n - 1]}{n^3\pi^3} - \frac{6(-1)^n}{n^5\pi^5}.$$

We get

$$u(x,t) = t^2 + x(1-t^2) + \sum_{n=1}^{\infty}\left[A_n e^{-n^2\pi^2 t} + \frac{6(-1)^{n+1}(n^2\pi^2 t - 1)}{n^5\pi^5}\right]\sin(n\pi x).$$