12.5.10. $y'' - 2xy' + 2ny = 0$

$$r(x) = e^{-\int 2x\,dx} = e^{-x^2}$$

$\Rightarrow$ self-adjoint form: $(e^{-x^2} y')' + 2n \underbrace{e^{-x^2}}_{\text{weight}} y = 0$

e^{-x^2} vanishes at $-\infty$ & $+\infty$

therefore $\int_{-\infty}^{+\infty} e^{-x^2} H_n(x) H_m(x) = 0$, if $n \neq m$.

13.1.6. $u = W(x)V(y)$

$$y W'(x) V(y) + x W(x) V'(y) = 0$$

$$\Rightarrow \frac{W'(x)}{xW(x)} + \frac{V'(y)}{yV(y)} = 0$$

$$\Rightarrow \begin{cases} W'(x) = 2k\,x\,W(x) \\ V'(y) = -2k\,y\,V(y) \end{cases} \quad \text{for some } k$$

$$\frac{W'}{W} = 2kx \Rightarrow W = e^{2\int kx\,dx} = c_1 e^{kx^2}$$

Similarly, $V = c_2 e^{-ky^2}$

$\Rightarrow u = c\, e^{k(x^2 - y^2)}$ c, k are constants.

14. $u = W(x)\,V(y)$

$\Rightarrow \; x^2 W''(x) V(y) + W(x) V''(y) = 0$

$\Rightarrow \; \dfrac{x^2 W''(x)}{W(x)} + \dfrac{V''(y)}{V(y)} = 0$

$\Rightarrow \begin{cases} x^2 W''(x) = k\,W(x) & (1) \\ V''(y) = -k\,V(y) & (2) \end{cases}$ for some k

(1) has solution $W = c_1 x^{\alpha_1} + c_2 x^{\alpha_2}$
where $\alpha_{1,2}$ are solutions of $\alpha(\alpha-1) = k$
i.e. $\alpha_{1,2} = \dfrac{1 \pm \sqrt{1+4k}}{2}$

If $k \neq -\frac{1}{4}$, $\alpha_1 \neq \alpha_2$ done.

If $k = -\frac{1}{4}$, $\alpha_1 = \alpha_2 = \frac{1}{2}$
$W = \left(c_1 + c_2 \ln|x|\right)\sqrt{x}$.

(2). has solution when $k=0$, $V = d_1 + d_2 y$
when $k > 0$ $V = d_1 \cos\sqrt{k}\,y + d_2 \sin\sqrt{k}\,y$
$k < 0$ $V = d_1 \cosh\sqrt{-k}\,y + d_2 \sinh\sqrt{-k}\,y$.

The full solution is then generated.