13.3.2. From $u(0,t)$ & $u(L,t)=0$ we have the solution for heat equation as

$$u = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{L}x\right) e^{-k\left(\frac{n\pi}{L}\right)^2 t}$$

At $t=0$ $\sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{L}x\right) = x(L-x)$

Hence $A_n = \frac{2}{L}\int_0^L x(L-x)\sin\left(\frac{n\pi}{L}x\right)dx$

$$= \frac{2}{n\pi}\int_0^{n\pi} \frac{L^2 x}{n\pi}\left(1-\frac{x}{n\pi}\right)\sin x \, dx$$

$$= \frac{2L^2}{(n\pi)^2}\Big[x\left(1-\frac{x}{n\pi}\right)(-\cos x) + \left(1-\frac{2x}{n\pi}\right)\sin x - \frac{2}{n\pi}\cos x \Big]\Big|_0^{n\pi}$$

$$= \frac{4L^2}{(n\pi)^3}\left[1-(-1)^n\right]$$

$$\Rightarrow \boxed{u = \frac{4L^2}{\pi^3}\sum_{n=1}^{\infty}\frac{1-(-1)^n}{n^3}\sin\left(\frac{n\pi}{L}x\right) e^{-k\left(\frac{n\pi}{L}\right)^2 t}.}$$

13.4.8 Wave equation with $\left.\frac{\partial u}{\partial x}\right|_{x=0} = 0 = \left.\frac{\partial u}{\partial x}\right|_{x=L}$

$$\Rightarrow \quad u = A_0 t + \frac{B_0}{2} + \sum_{n=1}^{\infty}\cos\left(\frac{n\pi}{L}x\right)\left[A_n \sin\left(\frac{an\pi}{L}t\right) + B_n\cos\left(\frac{an\pi}{L}t\right)\right]$$

since $\left.\frac{\partial u}{\partial t}\right|_{t=0} = 0 = A_0 + \sum_{n=1}^{\infty}\cos\left(\frac{n\pi}{L}x\right)\cdot A_n\cdot\frac{an\pi}{L}$

we have $A_i = 0$, $i=0,1,\ldots$

Therefore $u(x,0) = x = \frac{B_0}{2} + \sum_{n=1}^{\infty}\cos\left(\frac{n\pi}{L}x\right)\cdot B_n$

$$B_0 = \frac{2}{L}\int_0^L x\,dx = L.$$

$$B_n = \frac{2}{L}\int_0^L x \cdot \cos\left(\frac{n\pi}{L}x\right) dx$$

$$= \frac{2}{n\pi}\int_0^{n\pi} \frac{L}{n\pi}\, x \cos x \, dx$$

$$= \frac{2L}{(n\pi)^2}\left[x \sin x + \cos x\right]\Big|_0^{n\pi}$$

$$= \frac{2L}{(n\pi)^2}\left[(-1)^n - 1\right]$$

$$\Rightarrow \boxed{u = \frac{L}{2} + \frac{2L}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n - 1}{n^2}\cos\left(\frac{n\pi}{L}x\right)\cos\left(\frac{an\pi}{L}t\right).}$$

B.4.12. (a) $\xi = x + at$, $\quad \eta = x - at$

$$\partial_x = 1\cdot\partial_\xi + 1\cdot\partial_\eta, \quad \partial_t = a\cdot\partial_\xi - a\,\partial_\eta$$

$$\Rightarrow \partial_x^2 = \partial_\xi^2 + \partial_\eta^2 + 2\partial_\xi\partial_\eta, \quad \partial_t^2 = a^2\left(\partial_\xi^2 + \partial_\eta^2 - 2\partial_\xi\partial_\eta\right)$$

$$\left(a^2\partial_x^2 - \partial_t^2\right)u = 4a^2\,\partial_\xi\partial_\eta\, u = 0.$$

$$\Rightarrow \partial_\xi\partial_\eta u = 0$$

(b). $\partial_\xi(\partial_\eta u) = 0 \Rightarrow \partial_\eta u = g(\eta)$ for some g

$\Rightarrow u = G(\eta) + F(\xi)$. where $G' = g$.

$u(x,t) = F(x+at) + G(x-at)$

$$\begin{cases} u(x,0) = f(x) = F(x) + G(x) & \text{①} \\ \left.\frac{\partial u}{\partial t}\right|_{t=0} = g(x) = aF'(x) - aG'(x) & \text{②} \end{cases}$$

② $\Rightarrow$ $F(x) - G(x) = \int_{x_0}^{x}\frac{g(s)}{a}\,ds + 2C$ ③

for some x_0 & C

Combining with ① we have

$$F(x) = \frac{① + ③}{2} = \frac{f(x)}{2} + \frac{1}{2a}\int_{x_0}^{x} g(s)\,ds + c$$

$$G(x) = \frac{① - ③}{2} = \frac{f(x)}{2} - \frac{1}{2a}\int_{x_0}^{x} g(s)\,ds - c$$

(c). $u(x,t) = F(x+at) + G(x-at)$

$$= \frac{1}{2} f(x+at) + \frac{1}{2a}\int_{x_0}^{x+at} g(s)\,ds$$

$$+ \frac{1}{2} f(x-at) + \frac{1}{2a}\int_{x_0}^{x-at} g(s)\,ds$$

$$= \frac{1}{2}\left[f(x+at) + f(x-at)\right] + \frac{1}{2a}\int_{x-at}^{x+at} g(s)\,ds$$

13.5.2. Laplacian Equation with $u(0,y) = 0 = u(a,y)$

$$\Rightarrow \quad u = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi}{a}x\right)\left[A_n \sinh\left(\frac{n\pi}{a}y\right) + B_n \cosh\left(\frac{n\pi}{a}y\right)\right]$$

$$\left.\frac{\partial u}{\partial y}\right|_{y=0} = 0 = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi}{a}x\right) \cdot A_n \cdot \frac{n\pi}{a}$$

$$\Rightarrow A_n = 0$$

$$u(x,b) = f(x) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi}{a}x\right) \cdot B_n \cosh\left(\frac{n\pi}{a}b\right)$$

$$\Rightarrow \boxed{B_n = \frac{1}{\cosh\left(\frac{n\pi b}{a}\right)} \cdot \frac{2}{a}\int_0^a f(x) \cdot \sin\left(\frac{n\pi}{a}x\right) dx}$$