18.2 2. $f(z) = z^2 + \frac{1}{z-4}$

the only pole at $z=4$, outside $|z|=1$

6. $f(z) = e^z / (2z^2+11z+15)$

The poles are the solutions of $2z^2+11z+15$

$$z_{1,2} = \frac{-11 \pm \sqrt{11^2 - 8\cdot 15}}{4} = \frac{-11 \pm 3i}{4}$$

$$|z_1| = |z_2| = \frac{1}{4}\sqrt{11^2+3^2} = \sqrt{30}/2 > 1$$

Both of them are outside $|z|=1$

10. Only pole $z=-(1+i)$, inside the contour

$$\Rightarrow \oint_C \frac{5}{z+1+i}\,dz = 5\cdot 2\pi i = 10\pi i$$

18. The function has two poles, $z=-2$, $z=2i$

(a). Both of the poles are inside $|z|=5$.

$$\Rightarrow \oint_{|z|=5} \left(\frac{3}{z+2} - \frac{1}{z-2i}\right) dz = 3\cdot(2\pi i) - 2\pi i = 4\pi i$$

(b). Only $z=2i$ is inside $|z-2i|=1/2$

$$\Rightarrow \oint_{|z-2i|=1/2} \left(\underbrace{\frac{3}{z+2}}_{\text{no contribution}} - \frac{1}{z-2i}\right) dz = -2\pi i$$

20. $$\frac{1}{z^3+2iz^2} = \frac{1}{z^2}\cdot\frac{1}{z+2i} = \frac{1}{4}\left(\frac{z-2i}{z^2} - \frac{1}{z+2i}\right)$$

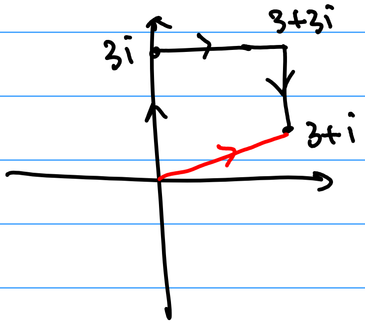
$$= \frac{1}{4}\left(\frac{1}{z} - \frac{2i}{z^2} - \underbrace{\frac{1}{z+2i}}_{\text{no contribution}}\right)$$

The poles are $z=0$, $-2i$. Only $z=0$ is inside $|z|=1$. Therefore

$$\oint_{|z|=1} \frac{1}{z^3+2iz^2}\,dz = \oint_{|z|=1} \frac{1}{4}\frac{1}{z}\,dz = \frac{1}{4}\cdot 2\pi i = \frac{\pi i}{2}$$

18.3 2. (a) We use the red path

$z = (3+i)t, \quad 0 \leq t \leq 1$



$$\int_C e^z \, dz = \int_0^1 e^{(3+i)t} (3+i) \, dt$$

$$= e^{(3+i)t} \Big|_0^1 = e^{3+i} - 1$$

(b) $$\int_C e^z \, dz = e^z \Big|_0^{3+i} = e^{3+i} - 1$$

4. $$\int_C 6z^2 \, dz, \quad z(t) = 2\cos^3 \pi t - i \sin^2 \frac{\pi}{4} t$$
$$0 \leq t \leq 2$$

As always, it's tedious to really calculate the integral along the indicated path. But since the integrand is analytic, it only depends on the end points:

$$\int_C 6z^2 \, dz = 2z^3 \Big|_{z(0)}^{z(1)} = 2\left(-2 - i \cdot \frac{1}{2}\right)^3 - 2(2)^3$$

$$= -2\left(2^3 + 3 \cdot 2^2 \cdot \frac{i}{2} + 3 \cdot 2 \cdot \frac{-1}{4} + \frac{-i}{8}\right) - 2 \cdot 2^3$$

$$= -61/2 - i \cdot \frac{47}{4}$$

8. $\int_{-3i}^{2i} (z^3 - z)\, dz = \frac{1}{4} z^4 - \frac{1}{2} z^2 \Big|_{3i}^{2i}$

$= \frac{1}{4}(2^4 - 3^4) + \frac{1}{2}(2^2 - 3^2) = -75/4$

20. $\int_{1-i}^{1+\sqrt{3}i} \left(\frac{1}{z} + \frac{1}{z^2} \right) dz$

$= \ln \frac{1+\sqrt{3}i}{1-i} - \frac{1}{z} \Big|_{1-i}^{1+\sqrt{3}i}$

$= \ln\sqrt{2} + i\left(\frac{\pi}{3} + \frac{\pi}{4}\right) - \frac{1}{1+\sqrt{3}i} + \frac{1}{1-i}$

$= \frac{1}{2}\ln 2 + \frac{1}{4} + i\left(\frac{7\pi}{12} + \frac{\sqrt{3}}{4} + \frac{1}{2}\right)$

22. $\int_0^i z\sin z\, dz = -z\cos z + \sin z \Big|_0^i$

$= -i\cos i + \sin i - 0$

$= -i\cosh 1 + i\sinh 1$ $\qquad \frac{e^1 - e^{-1}}{2} - \frac{e^1 + e^{-1}}{2}$

$= -ie^{-1}$