19.3.14 $\quad |z| > 2$

$$\frac{1}{(z-1)(z-2)} = \frac{1}{z-2} - \frac{1}{z-1}$$

$$= \frac{1}{z}\left(\frac{1}{1-2/z} - \frac{1}{1-1/z}\right)$$

$$= \frac{1}{z} \sum_{n=0}^{\infty} \left(\frac{2}{z}\right)^n - \left(\frac{1}{z}\right)^n$$

$$= \sum_{n=0}^{\infty} \left(2^n - 1\right) z^{-n-1}$$

16. $0 < |z-2| < 1$

$$\frac{1}{(z-1)(z-2)}$$

$$= \frac{1}{z-2} \cdot \frac{1}{1+z-2} = \frac{1}{z-2} \cdot \sum_{n=0}^{\infty} (-1)^n (z-2)^n$$

$$= \sum_{n=0}^{\infty} (-1)^n (z-2)^{n-1}$$

22. $|z| > 1$

$$\frac{1}{z(1-z)^2} = \frac{1}{z}\, \partial_z \frac{1}{1-z}$$

$$= -\frac{1}{z}\, \partial_z\, \frac{1}{z}\, \frac{1}{1-1/z}$$

$$= -\frac{1}{z}\, \partial_z\, \frac{1}{z} \sum_{n=0}^{\infty} z^{-n}$$

$$= -\frac{1}{z}\, \partial_z \sum_{n=0}^{\infty} z^{-n-1}$$

$$= -\frac{1}{z} \sum_{n=0}^{\infty} (-n-1)\, z^{-n-2}$$

$$= \sum_{n=0}^{\infty} (n+1)\, z^{-n-3}$$

19.4.2.

$$\sin 4z = \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} (4z)^{2n+1}$$

$$\sin 4z - 4z = \sum_{n=1}^{\infty} \frac{1}{(2n+1)!} (4z)^{2n+1}$$

$$= \sum_{n=0}^{\infty} \frac{1}{(2n+3)!} (4z)^{2n+3}$$

$$\frac{\sin 4z - 4z}{z^2} = \sum_{n=0}^{\infty} \frac{4^{2n+3}}{(2n+3)!}\, z^{2n+1}$$

no principal part $\Rightarrow$ removable

$f(0) = 0.$

8. $\sin z = 0$, $z = n\pi$. n integer

$\sin' z = \cos z$. Since $\cos n\pi \neq 0$.

$n\pi$ is a zero of order 1

$\Rightarrow$ $\sin^2 z = 0$ has zero of order 2

at $z = n\pi$

14. $f(z) = 5 - 6/z^2$

pole: 0. order: 2

18. $f(z) = \dfrac{\cot \pi z}{z^2} = \dfrac{\cos \pi z}{z^2 \sin \pi z}$

poles: $z = 0$ or $\sin \pi z = 0$, $z = n$.

order: $2 + 1 = 3$ at $z = 0$

1 at $z = n \neq 0$.

17.5.8. $f(z) = \dfrac{4z+8}{2z-1} = \dfrac{2z+4}{z-1/2}$

pole of order 1 at $z = 1/2$

$\mathrm{Res}(f, 1/2) = 2z+4 \Big|_{z=1/2} = 5$

10. $f(z) = \dfrac{1}{(z^2-2z+2)^2}$

poles of order 2 at $z = 1 \pm i$

Let $z_1 = 1+i$, $z_2 = 1-i$ $\quad z_1 - z_2 = 2i$

$f(z) = \dfrac{1}{(z-z_1)^2} \dfrac{1}{(z-z_2)^2} = \dfrac{1}{(z-z_1)^2} \cdot \dfrac{1}{(z-z_1+z_1-z_2)^2}$

$= \dfrac{1}{(z-z_1)^2} \cdot \dfrac{1}{(z_1-z_2)^2} \left(1 - \dfrac{2(z-z_1)}{z_1-z_2} + \cdots \right)$

$\mathrm{Res}(f, z_1) = -\dfrac{2}{(z_1-z_2)^3} = -\dfrac{i}{4}$

$\Rightarrow \mathrm{Res}(f, z_2) = i/4$ by conjugation.

22. $$\oint_C \frac{1}{z^3(z-1)^4}\,dz = 2\pi i \,\mathrm{Res}\left(\frac{1}{z^3(z-1)^4}, 1\right)$$

$C: |z-2| = 3/2$

1 is a pole of order 4

$$\Rightarrow \frac{1}{z^3} = \frac{1}{(z-1+1)^3}$$

$$\left(\text{Use } \frac{1}{(1-z)^m} = \frac{1}{(m-1)!}\partial_z^{m-1} \frac{1}{1-z} = \sum_{n=m-1}^{\infty} \binom{n}{m-1} z^{n-(m-1)} \right)$$

$$= \sum_{n=2}^{\infty} \binom{n}{2} (1-z)^{n-2}$$

We want $n=5$ term which is

$$\frac{5\cdot 4}{2}(1-z)^3 = -10(z-1)^3$$

$$\Rightarrow \mathrm{Res}\left(\frac{1}{z^3(z-1)^4}, 1\right) = -10$$

$$\Rightarrow \oint_C \frac{1}{z^3(z-1)^4}\,dz = -20\pi i$$

28. $$\oint_C \frac{\cot \pi z}{z^2}\, dz = 2\pi i \operatorname{Res}\left(\frac{\cot \pi z}{z^2}, 0\right)$$

$C: |z| = 1/2$

0 is a pole of order 3

$\Rightarrow$ In $\cot \pi z = \dfrac{\cos \pi z}{\sin \pi z}$, we check the 3rd term in Laurent expansion

$$\cos \pi z = 1 - \frac{1}{2}(\pi z)^2 + \cdots$$

$$\sin \pi z = \pi z \left[1 - \frac{1}{6}(\pi z)^2 + \cdots\right]$$

$$\Rightarrow \cot \pi z = \frac{1}{\pi z}\left[1 + \left(\frac{1}{6} - \frac{1}{2}\right)(\pi z)^2 + \cdots\right]$$

$$\operatorname{Res}\left(\frac{\cot \pi z}{z^2}, 0\right) = -\frac{1}{3}\pi$$

$$\Rightarrow \oint_C \frac{\cot \pi z}{z^2}\, dz = -\frac{2\pi^2}{3} i$$

19.6.6 $\displaystyle\int_0^{\pi} \frac{1}{1+\sin^2\theta}\,d\theta = \int_0^{\pi} \frac{2}{2+1-\cos 2\theta}\,d\theta$

$$= \int_0^{2\pi} \frac{1}{3-\cos\theta}\,d\theta$$

$$= \oint_{|z|=1} \frac{1}{3-\frac{z+z^{-1}}{2}}\,\frac{dz}{iz}$$

$$= 2i \oint_{|z|=1} \frac{1}{\underbrace{z^2-6z+1}}\,dz \Rightarrow z^2-6z+1=0$$

$$z = \frac{6 \pm \sqrt{32}}{2}$$

$$= 2i \cdot 2\pi i \,\mathrm{Res}\left(\frac{1}{z^2-6z+1},\ 3-\sqrt{8}\right) \qquad = 3 \pm \sqrt{8}$$

Only $3-\sqrt{8}$ is inside $|z|=1$

$$= -4\pi \cdot \frac{1}{(3-\sqrt{8})-(3+\sqrt{8})}$$

$$= \frac{4\pi}{2\sqrt{8}} = \frac{\pi}{\sqrt{2}}$$

For 19.6.5: First $\displaystyle\int_0^{\pi} \frac{1}{2-\cos\theta}\,d\theta = \int_{\pi}^{2\pi} \frac{1}{2-\cos t}\,dt$

$$= \frac{1}{2} \underbrace{\int_0^{2\pi} \frac{1}{2-\cos\theta}\,d\theta}.$$

exactly as 19.6.6.

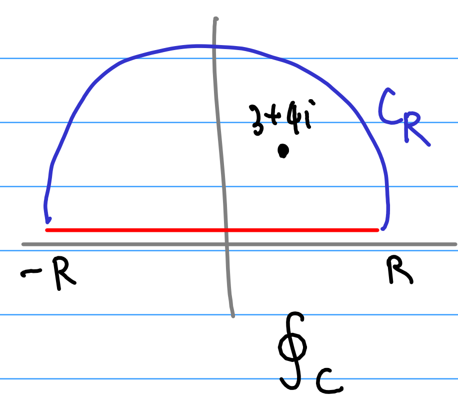
The answer, if you follow the procedure in 19.6.6, should be $\boxed{-2\pi / z_- - z_+}$ $z_\pm$ are roots of $z^2 - 4z + 1 = 0$. $z_\pm = 2 \pm \sqrt{3}$, ans $= \boxed{\pi / \sqrt{3}}$.

12. $\dfrac{1}{z^2 - 6z + 25}$ has pole at $z = 3 \pm 4i$

Since $z^2 - 6z + 25$ has deg 2.

$$\lim_{R \to \infty} \int_{C_R} \frac{1}{z^2 - 6z + 25} = 0 \quad \text{if } C_R \text{ is part of } |z| = R$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{1}{z^2 - 6z + 25}\, dz$$



$$= \lim_{R \to \infty} \int_{-R}^{R} \frac{1}{z^2 - 6z + 25}\, dz$$

$$= \lim_{R \to \infty} \oint_C \frac{1}{z^2 - 6z + 25}\, dz$$

$$= 2\pi i \operatorname{Res}\left(\frac{1}{z - 6z + 25},\ 3 + 4i \right)$$

$$= 2\pi i \frac{1}{3 + 4i - (3 - 4i)} = \pi / 4$$

20. $\dfrac{1}{z^6+1}$ has poles at $z_p = e^{i\pi/6\,(1+2p)}$
for $p = 0, 1, \dots, 5$

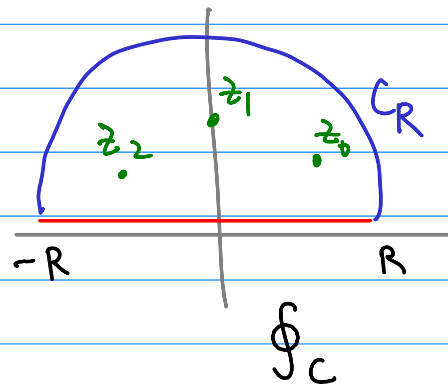
Again since z^6+1 has deg $6 \geq 2$

$$\lim_{R\to\infty} \int_{C_R} \frac{1}{z^6+1}\, dz = 0$$ if C_R is part of $|z| = R$.

$$\Rightarrow \int_0^\infty \frac{1}{z^6+1}\, dz = \lim_{R\to\infty} \int_0^R \frac{1}{z^6+1}\, dz$$

$$= \lim_{R\to\infty} \frac{1}{2} \int_{-R}^{R} \frac{1}{z^6+1}\, dz$$

$$= \frac{1}{2} \oint_C \frac{1}{z^6+1}\, dz$$



$$= \frac{1}{2}\, 2\pi i \cdot \left[\mathrm{Res}\left(\frac{1}{z^6+1}, z_0\right) + \mathrm{Res}\left(\frac{1}{z^6+1}, z_1\right) + \mathrm{Res}\left(\frac{1}{z^6+1}, z_2\right) \right]$$

$$= \pi i \left(\frac{1}{6z_0^5} + \frac{1}{6z_1^5} + \frac{1}{6z_2^5} \right)$$

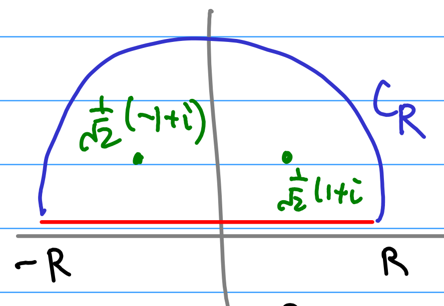
$$= \frac{\pi i}{-6} (z_0 + z_1 + z_2) = -\frac{\pi i}{6} \cdot 2i = \pi/3.$$

pf. $\dfrac{x\sin x}{x^4+1} = \operatorname{Im}\left(\dfrac{x e^{ix}}{x^4+1}\right)$.

$$\int_0^\infty \frac{x\sin x}{x^4+1}\,dx = \lim_{R\to\infty}\int_0^R \frac{x\sin x}{x^4+1}\,dx$$

$$= \lim_{R\to\infty}\frac{1}{2}\int_{-R}^{R}\frac{x\sin x}{x^4+1}\,dx$$

$$= \frac{1}{2}\operatorname{Im}\lim_{R\to\infty}\int_{-R}^{R}\underline{\frac{x e^{ix}}{x^4+1}}\,dx$$



$\oint_C$

Theorem 19.16 : use the upper half circle

Thm 19.16 : $\displaystyle\lim_{R\to\infty}\int_{C_R}\frac{z e^{iz}}{z^4+1}\,dz = 0$ if C_R is upper half of $|z|=R$

$$\Rightarrow \lim_{R\to\infty}\int_{-R}^{R}\frac{z e^{iz}}{z^4+1}\,dz = \oint_C \frac{z e^{iz}}{z^4+1}\,dz$$

$$= 2\pi i\left[\operatorname{Res}\left(\frac{z e^{iz}}{z^4+1}, \frac{1}{\sqrt{2}}+\frac{i}{\sqrt{2}}\right) + \operatorname{Res}\left(\frac{z e^{iz}}{z^4+1}, \frac{-1+i}{\sqrt{2}}\right)\right]$$

$$= 2\pi i\left(\left.\frac{z e^{iz}}{4z^3}\right|_{z=\frac{1}{\sqrt{2}}+\frac{i}{\sqrt{2}}} + \left.\frac{z e^{iz}}{4z^3}\right|_{z=-\frac{1}{\sqrt{2}}+\frac{i}{\sqrt{2}}}\right)$$

$$= \pi i\, e^{-1/\sqrt{2}}\sin\frac{1}{\sqrt{2}}$$

$$\Rightarrow \int_0^\infty \frac{x\sin x}{x^4+1}\,dx = \frac{\pi}{2}e^{-1/\sqrt{2}}\sin 1/\sqrt{2}.$$