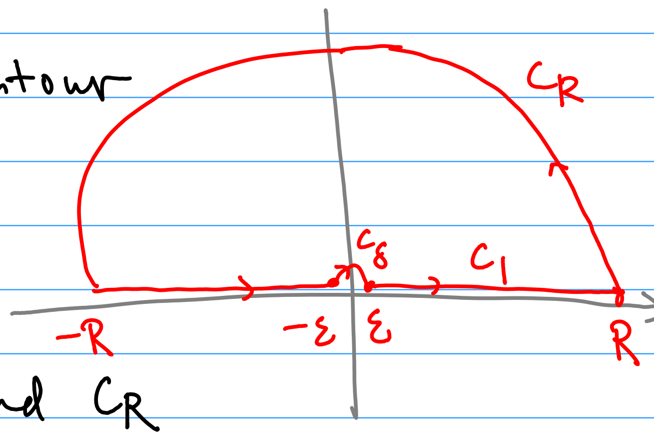
19.6.

32. $\displaystyle P.V. \int_{-\infty}^{\infty} \frac{\sin x}{x(x^2+1)}\, dx = \pi(1-e^{-1})$

To show this, use the contour that's made of four segments:



$(-R,-\varepsilon)$, C_δ, (ε, R) and C_R

We call $(-R,-\varepsilon)$ and (ε, R) C_1

$$\oint_C = \int_{C_1} + \int_{C_R} + \int_{C_\delta}$$

The next step: $\sin(x) = \mathrm{Im}(e^{ix})$

$$\Rightarrow \quad P.V. \int_{-\infty}^{\infty} \frac{\sin x}{x(x^2+1)}\, dx = \mathrm{Im}\Bigg(\underbrace{P.V. \int_{-\infty}^{\infty} \frac{e^{ix}}{x(x^2+1)}\, dx}_{\text{want to calculate 1st.}}\Bigg)$$

Function $\dfrac{e^{iz}}{z(z^2+1)}$ has a pole $z=i$ inside C. $\Rightarrow$

$$\oint_C \frac{e^{iz}}{z(z^2+1)}\, dz = 2\pi i \,\mathrm{Res}\left(\frac{e^{i}}{z(z^2+1)}, i\right)$$

$$= 2\pi i \cdot \left.\frac{e^{iz}}{z(z+i)}\right|_{z=i} = -\pi i \cdot e^{-1}$$

On the other hand

C_δ has only half of the circle

$$\lim_{\delta\to 0}\int_{C_\delta}\frac{e^{iz}}{z(z^2+1)}\,dz = -\frac{1}{2}\cdot \operatorname{Res}\left(\frac{e^{iz}}{z(z^2+1)},\,0\right)\cdot 2\pi i$$

C_δ is clockwise

$$= -\pi i$$

And

$$\lim_{R\to\infty}\left|\int_{C_R}\frac{e^{iz}}{z(z^2+1)}\,dz\right| \leq \lim_{R\to\infty}\cdot\frac{1}{R\cdot(R^2-1)}\cdot \pi R = 0$$

$$\Rightarrow \lim_{\substack{\delta\to 0\\ R\to\infty}}\left(\int_{C_1}+\int_{C_\delta}+\int_{C_R}\right)\frac{e^{iz}}{z(z^2+1)}\,dz$$

$$= \text{P.V.}\int_{-\infty}^{\infty}\frac{e^{ix}}{x(x^2+1)}\,dx - \pi i + 0$$

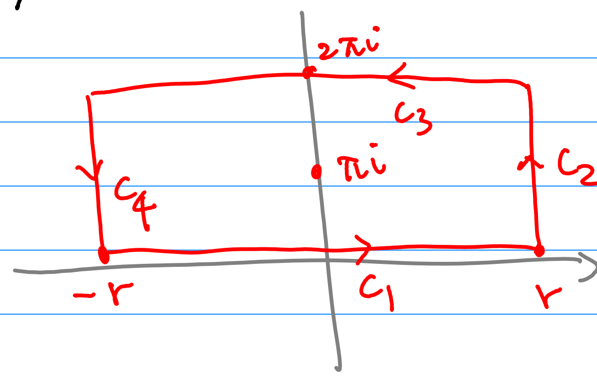
which should equal $\oint_C \frac{e^{iz}}{z(z^2+1)}\,dz = -e^{-1}\pi i$

$$\Rightarrow \text{P.V.}\int_{-\infty}^{\infty}\frac{e^{ix}}{x(x^2+1)}\,dx = \pi i\left(1-e^{-1}\right)$$

$$\Rightarrow \text{P.V.}\int_{-\infty}^{\infty}\frac{\sin x}{x(x^2+1)}\,dx = \pi\left(1-e^{-1}\right)$$

85. $\displaystyle P.V. \int_{-\infty}^{\infty} \frac{e^{ax}}{1+e^{x}}\,dx = \frac{\pi}{\sin ax}$, $0 < a < 1$

As hinted, we use the contour shown here.



$$\oint_C = \int_{C_1} + \int_{C_2} + \int_{C_3} + \int_{C_4}$$

For function $\dfrac{e^{az}}{1+e^{z}}$ $(0<a<1)$

the possible poles are at $1+e^{z}=0$,
which means $z=(2n+1)\pi i$. n integer.

Inside our contour, we have one πi.

$$\Rightarrow \oint_C \frac{e^{az}}{1+e^{z}}\,dz = 2\pi i \cdot \text{Res}\left(\frac{e^{az}}{1+e^{z}}, \pi i\right)$$

$$= -2\pi i \cdot e^{a\pi i}$$

Now $\displaystyle \int_{C_1} \frac{e^{az}}{1+e^{z}}\,dz = \int_{-r}^{r} \frac{e^{ax}}{1+e^{x}}\,dx$

and $\displaystyle \int_{C_3} \frac{e^{az}}{1+e^{z}}\,dz = \int_{r}^{-r} \frac{e^{a(x+2\pi i)}}{1+e^{(x+2\pi i)}}\,dx$

$$= -\int_{-r}^{r} \frac{e^{2a\pi i}\cdot e^{ax}}{1+e^{x}}\,dx = -e^{2a\pi i}\int_{-r}^{r} \frac{e^{ax}}{1+e^{x}}\,dx$$

Moreover,

$$\lim_{r\to\infty}\left|\int_{C_2}\frac{e^{az}}{1+e^z}\,dz\right| = \lim_{r\to\infty}\left|\int_0^{2\pi}\frac{e^{a(r+yi)}}{1+e^{r+yi}}\,i\,dy\right|$$

$$\leq \lim_{r\to\infty} 2\pi\cdot\frac{e^{ar}}{e^r-1} = 0.$$

$$\lim_{r\to\infty}\left|\int_{C_4}\frac{e^{az}}{1+e^z}\,dz\right| = \lim_{r\to\infty}\left|\int_{2\pi}^{0}\frac{e^{a(-r+yi)}}{1+e^{-r+yi}}\,i\,dy\right|$$

$$\leq \lim_{r\to\infty} 2\pi\cdot\frac{e^{-ar}}{1-e^{-r}} = 0$$

Therefore

$$\lim_{r\to\infty}\int_{C_1}+\int_{C_3}\frac{e^{az}}{1+e^z}\,dz = \oint_C\frac{e^{az}}{1+e^z}\,dz = -2\pi i\,e^{a\pi i}$$

Which means

$$\left(1-e^{2a\pi i}\right)\int_{-\infty}^{\infty}\frac{e^{ax}}{1+e^x}\,dx = -2\pi i\,e^{a\pi i}$$

$$\Rightarrow \int_{-\infty}^{\infty}\frac{e^{ax}}{1+e^x}\,dx = \frac{\pi}{\sin a\pi}$$

12.1.2 $f_1 = x^3$, $f_2 = x^2+1$

$$(f_1, f_2) = \int_{-1}^{1} x^3 \cdot (x^2+1)\, dx = \frac{x^6}{6} + \frac{x^4}{4} \Big|_{-1}^{1} = 0 .$$

12.1.8. $$\int_0^{\pi/2} \cos(2n+1)x \cos(2m+1)x \; dx$$

$$= \frac{1}{2}\int_0^{\pi/2} \left[\cos 2(n-m)x - \cos 2(n+m+1)x\right] dx$$

$$= \begin{cases} \frac{1}{2} \cdot \frac{\pi}{2} = \frac{\pi}{4} & \text{if } n = m \\ \frac{1}{2}\left[\frac{1}{2(n-m)} \sin 2(n-m)x \Big|_0^{\pi/2} - \frac{1}{2(n+m+1)} \sin 2(n+m+1)x \Big|_0^{\pi/2}\right] = 0 & \text{if } n \neq m \end{cases}$$

$$\Rightarrow \left(\cos(2n+1)x, \cos(2m+1)x\right) = \delta_{mn} \cdot \frac{\pi}{4} .$$

12.2.2 $$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\, dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} (-\cos nx)\, dx + \int_0^{\pi} 2\cos nx\, dx\right]$$

$$= \begin{cases} 1 & \text{if } n=0 \\ \frac{1}{\pi}\left[\left(-\frac{1}{n}\right)\sin nx\Big|_{-\pi}^{0} + \frac{2}{n}\sin nx\Big|_{0}^{\pi}\right] = 0 & \text{if } n \neq 0 \end{cases}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\, dx$$

$$= \frac{1}{\pi}\left[\frac{1}{n}\cos nx\Big|_{-\pi}^{0} - \frac{2}{n}\cos nx\Big|_{0}^{\pi}\right]$$

$$= \frac{1}{\pi}\left(\frac{1}{n}\cdot 2 - \frac{2}{n}(-2)\right) = 6/n\pi.$$

$$\Rightarrow f(x) = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{6}{n\pi}\sin nx$$

12.2.6

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

$$x^2 e^{ix} = \left(-i x^2 e^{ix} + 2x e^{ix} + 2i\, e^{ix}\right)'$$

$$= \frac{1}{\pi}\left(\int_{-\pi}^{\pi} \pi^2 \cos nx\,dx - \int_0^{\pi} x^2 \cos nx\,dx\right.$$

$$= \begin{cases} \dfrac{1}{\pi}\left(\pi^2\cdot 2\pi - \dfrac{\pi^3}{3}\right) = \dfrac{5}{3}\pi^2 & n=0 \\ -\dfrac{1}{n^3\pi}\left(n^2x^2\sin nx + 2nx\cos nx - 2\sin nx\right)\Big|_0^{\pi} & \\ = -\dfrac{1}{n^3\pi}\,2n\pi(-1)^n = \dfrac{2(-1)^{n+1}}{n^2} & n\neq 0 \end{cases}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

$$= \frac{1}{\pi}\left(\int_{-\pi}^{\pi} \pi^2 \sin nx\,dx - \int_0^{\pi} x^2 \sin nx\,dx\right.$$

$$= -\frac{1}{n^3\pi}\left(-n^2x^2\cos nx + 2nx\sin nx + 2\cos nx\right)\Big|_0^{\pi}$$

$$= -\frac{1}{n^3\pi}\left(-n^2\pi^2(-1)^n + 2\cdot(-1)^n - 2\right) = \frac{1}{n^3\pi}\left((n^2\pi^2-2)(-1)^n + 2\right)$$

$$xe^{ix} = \left(-ixe^{ix} + e^{ix}\right)'$$

12. $$a_n = \frac{1}{2}\int_{-2}^{2} f(x)\cos\frac{n\pi x}{2}\,dx$$

$$= \frac{1}{2}\left[\int_0^1 x\cos\frac{n\pi x}{2}\,dx + \int_1^2 \cos\frac{n\pi x}{2}\,dx\right]$$

$$= \begin{cases} \frac{1}{2}\cdot\left(\frac{1}{2}+1\right) = 3/4 & n=0 \\ \frac{1}{2}\cdot\frac{4}{n^2\pi^2}\cdot\left(\frac{n\pi}{2}x\sin\frac{n\pi x}{2} + \cos\frac{n\pi x}{2}\right)\Big|_0^1 + \frac{1}{2}\frac{2}{n\pi}\sin\frac{n\pi x}{2}\Big|_1^2 & n\neq 0 \end{cases}$$

$$e^{in\pi/2} = (i)^n = \begin{cases} \underbrace{(-1)^k}_{\cos} & n=2k \\ \underbrace{(-1)^k\cdot i}_{\sin} & n=2k+1 \end{cases}$$

$$= \frac{2}{n^2\pi^2}\left(\frac{n\pi}{2}\sin\frac{n\pi}{2} + \cos\frac{n\pi}{2} - 1\right) + \frac{1}{n\pi}\left(-\sin\frac{n\pi}{2}\right]$$

$$a_{2k\neq 0} = \frac{1}{2k^2\pi^2}\left((-1)^k - 1\right)$$

$$a_{2k+1} = \frac{2}{(2k+1)^2\pi^2}\left(\frac{2k+1}{2}\pi\cdot(-1)^k - 1\right) + \frac{(-1)^{k+1}}{(2k+1)\pi}.$$

$$b_n = \frac{1}{2}\int_{-2}^{2} f(x)\sin\frac{n\pi x}{2}\,dx$$

$$= \frac{1}{2}\left[\int_0^1 x\sin\frac{n\pi x}{2}\,dx + \int_1^2 \sin\frac{n\pi x}{2}\,dx\right]$$

$$= \frac{1}{2} \cdot \frac{4}{n^2\pi^2} \cdot \left(-\frac{n\pi}{2} x \cos\frac{n\pi x}{2} + \sin\frac{n\pi x}{2} \right) \Bigg|_0^1$$

$$- \frac{1}{2} \frac{2}{n\pi} \cos\frac{n\pi x}{2} \Bigg|_1^2$$

$$= \frac{2}{n^2\pi^2} \left(-\frac{n\pi}{2} \cos\frac{n\pi}{2} + \sin\frac{n\pi}{2} \right) - \frac{1}{n\pi}\left[(-1)^n - \cos\frac{n\pi}{2} \right]$$

$$b_{2k} = \frac{1}{2k^2\pi^2} \left(-k\pi (-1)^k \right) - \frac{1}{2k\pi} \left(1 - (-1)^k \right)$$

$$b_{2k+1} = \frac{2}{(2k+1)^2\pi^2} (-1)^k + \frac{1}{(2k+1)\pi}.$$

24. from 9. we get

$$\sin x = \frac{1}{2} \sin x + \frac{1}{\pi} + \frac{1}{\pi} \sum_{n=2}^{\infty} \frac{1+(-1)^n}{1-n^2} \cos nx$$

$$(0 \le x < \pi)$$

Pick $x = \pi/2$

$$1 = \frac{1}{2} + \frac{1}{\pi} + \frac{1}{\pi} \sum_{k=1}^{\infty} \frac{2}{(1-2k)(1+2k)} (-1)^k$$

$$\Rightarrow \frac{\pi}{4} = \frac{1}{2} + \sum_{k=1}^{\infty} \frac{(-1)^k}{(1-2k)(1+2k)}$$

12.4.2. $f(x) = \sum_{n=-\infty}^{\infty} c_n e^{in\pi x}$

$$c_n = \frac{1}{2}\int_0^2 f(x) e^{-in\pi x}\, dx$$

$$= \frac{1}{2}\int_1^2 e^{-in\pi x}\, dx$$

$$= \begin{cases} \frac{1}{2} & n=0 \\ \frac{1}{2}\frac{1}{-in\pi} e^{-in\pi x}\Big|_1^2 & n\neq 0 \end{cases}$$

$$= \frac{i}{2n\pi}\left(e^{-2n\pi i} - e^{-in\pi}\right) = \frac{i}{2n\pi}\left[1-(-1)^n\right]$$

10. $f(x) = \begin{cases} \cos x = \frac{e^{ix}+e^{-ix}}{2} & 0<x<\pi/2 \\ 0 & \pi/2 < x < \pi \end{cases}$

$$c_n = \frac{1}{\pi}\int_0^{\pi/2} \frac{e^{ix}+e^{-ix}}{2} e^{-i2nx}\, dx$$

$$= \frac{1}{2\pi}\left[\frac{1}{i(1-2n)} e^{i(1-2n)x}\Big|_0^{\pi/2} - \frac{1}{i(1+2n)} e^{-i(1+2n)x}\Big|_0^{\pi/2}\right]$$

$$= \frac{1}{2\pi i}\left[\frac{1}{1-2n}\left(i^{1-2n}-1\right) - \frac{1}{1+2n}\left(i^{-1-2n}-1\right)\right]$$

$$= \frac{1}{2\pi}\left[\frac{1}{1-2n}\left((-1)^n + i\right) + \frac{1}{1+2n}\left((-1)^n - i\right)\right]$$

$$= \frac{(-1)^n + i \cdot 2n}{\pi\left(1-4n^2\right)}$$