12.3 2. x is odd $\cos x$ is even

$x \cos x$ is odd.

4. odd

6. even

8. even

12 $f(x)$ is even. $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(n \cdot \frac{\pi}{2} \cdot x\right)$

$$a_n = \frac{1}{2} \int_{-2}^{2} f(x) \cos\left(n \cdot \frac{\pi}{2} x\right) dx$$

$$= \frac{1}{2} \int_{1}^{2} \cos\left(\frac{n\pi x}{2}\right) dx$$

$$= \frac{2}{n\pi} \sin \frac{n\pi x}{2} \Big|_{1}^{2}$$

$$= -\frac{2}{n\pi} \sin \frac{n\pi}{2}$$

$$\sin \frac{n\pi}{2} = \begin{cases} 0 & n \text{ even} \\ (-1)^k & n = 2k+1 \end{cases}$$

$$\Rightarrow f(x) = -\sum_{k=0}^{\infty} \frac{2}{(2k+1)\pi} (-1)^k \cos \frac{(2k+1)\pi x}{2}$$

18. $f(x)$ is odd

$$f(x) = \sum_{n=1}^{\infty} b_n \cdot \sin nx .$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx \, dx$$

$$= \frac{2}{\pi}\int_{0}^{\pi} f(x)\sin nx \, dx$$

$$= \frac{2}{\pi}\int_{0}^{\pi} x^3 \sin nx \, dx$$

$$= \frac{2}{n^4\pi}\left(-n^3x^3\cos nx + 3n^2x^2\sin nx + 6nx\cos nx - 6\sin nx\right)\Big|_0^{\pi}$$

$$= \frac{2}{n^4\pi}\left[-n^3\pi^3(-1)^n + 6n\pi(-1)^n\right] = \frac{2(-1)^n}{n^3}\left(-n^2\pi^2+6\right)$$

$$e^{ix} = \left(-i e^{ix}\right)'$$

$$xe^{ix} = \left(-ixe^{ix} + e^{ix}\right)'$$

$$x^2e^{ix} = \left(-ix^2e^{ix} + 2xe^{ix} + 2ie^{ix}\right)'$$

$$x^3e^{ix} = \left(-ix^3e^{ix} + 3x^2e^{ix} + i\cdot 6xe^{ix} - 6e^{ix}\right)'$$

$$x^3\sin x = \left(-x^3\cos x + 3x^2\sin x + 6x\cos x - 6\sin x\right)'$$

24. $f(x)$ is even. $\quad f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos 2nx$

$$a_n = \frac{2}{\pi}\int_{-\pi/2}^{\pi/2} f(x)\cos 2nx \, dx$$

$$= \frac{4}{\pi}\int_{0}^{\pi/2} \cos x \cos 2nx \, dx$$

$$= \frac{2}{\pi}\int_{0}^{\pi/2}\left[\cos(2n-1)x + \cos(2n+1)x\right] dx$$

$$= \frac{2}{\pi}\left(\frac{1}{2n-1} + \frac{1}{2n+1}\right) = \frac{8n}{\pi(4n^2-1)} .$$

26. For even. $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\pi x$

$$a_n = 2\int_0^1 f(x)\cos n\pi x \, dx$$

$$= 2\int_{1/2}^1 \cos n\pi x \, dx$$

$$= \begin{cases} 1 & n=0 \\ \frac{2}{n\pi}\sin n\pi x \Big|_{1/2}^1 = -\frac{2}{n\pi}\sin\frac{n\pi}{2} = \begin{cases} 0 & n=2k\neq 0 \\ \frac{2}{n\pi}(-1)^{k+1} & n=2k+1 \end{cases} \end{cases}$$

$$\Rightarrow f(x) = \frac{1}{2} + \sum_{k=0}^{\infty} \frac{2}{n\pi}(-1)^{k+1} \cdot \cos(2k+1)\pi x$$

For odd $f(x) = \sum_{n=1}^{\infty} b_n \sin n\pi x$

$$b_n = 2\int_0^1 f(x)\sin n\pi x \, dx$$

$$= 2\int_{1/2}^1 \sin n\pi x \, dx$$

$$= -\frac{2}{n\pi}\cos n\pi x \Big|_{1/2}^1$$

$$\cos\frac{n\pi}{2} = \begin{cases} (-1)^k & n=2k \\ 0 & n \text{ odd} \end{cases}$$

$$= \begin{cases} -\frac{2}{n\pi}\left[(-1)^{2k} - (-1)^k\right] = -\frac{2}{n\pi}\left(1-(-1)^k\right) & n=2k \\ -\frac{2}{n\pi}(-1)^{2k+1} = \frac{2}{n\pi} & n=2k-1 \end{cases}$$

$$\Rightarrow f(x) = \sum_{k=1}^{\infty}\left\{\left[\frac{2}{(2k-1)\pi}\sin(2k-1)\pi x - \frac{1-(-1)^k}{k\pi}\sin(2k\pi x)\right]\right.$$

12.5.2. (1) $\lambda = 0$. $y = a + bx$.

$y' = b$.

$y(1) = 0 \Rightarrow a + b = 0$

$y(0) + y'(0) = 0 \Rightarrow a + b = 0$

$\Rightarrow b = -a$

$y = a(1-x)$

(2) $\lambda = \alpha^2$ $y = a\cos\alpha x + b\sin\alpha x$

$y' = -a\alpha\sin\alpha x + b\alpha\cos\alpha x$

$y(1) = 0 \Rightarrow a\cos\alpha + b\sin\alpha = 0$

$y(0) + y'(0) = 0 \Rightarrow a + b\alpha = 0$

$\Downarrow$

$a = -b\alpha$

$\Rightarrow \alpha\cos\alpha = \sin\alpha$

$\Downarrow$

$\boxed{\alpha = \tan\alpha}$

$$y = b\left(-\alpha\cos\alpha x + \sin\alpha x\right)$$

(3) $\lambda = -\alpha^2$ $y = a\cosh\alpha x + b\sinh\alpha x$

$y' = a\alpha\sinh\alpha x + b\alpha\cosh\alpha x$

$y(1)=0 \Rightarrow \quad a\cosh\alpha + b\sinh\alpha = 0$

$y'(0)+y(0)=0 \Rightarrow \quad a + b\alpha = 0$

$\Downarrow$

$a=-b\alpha$

$\Rightarrow \alpha\cosh\alpha = \sinh\alpha$

$\Downarrow$

$\boxed{\alpha = \tanh\alpha}$

The equation $\alpha = \tanh\alpha$ has only one solution which is $\alpha = 0$. Therefore, this case doesn't provide any nontrivial solutions.