MATH 104 HW 2

CLAY SHONKWILER

§5.1

5. Find the area of the shaded region (pictured in the textbook)

Answer: Note that both curves are symmetric, so we can simply find the area of the righthand region and multiply by 2. The curve $y = 2x^2$ is above the curve $y = x^4 - 2x^2$ throughout the region. The points of intersection are given, so the area of the righthand region is given by

$$\begin{aligned} A &= \int_0^2 \left[2x^2 - (x^4 - 2x^2)\right] dx \\ &= \int_0^2 \left[4x^2 - x^4\right] dx \\ &= \frac{4}{3}x^3 - \frac{x^5}{5}\Big]_0^2 \\ &= \left(\frac{4}{3}(8) - \frac{32}{5}\right) - 0 \\ &= \frac{32}{3} - \frac{32}{5} \\ &= \frac{64}{15}. \end{aligned}$$

Hence, the area of the whole region is $2 \cdot \frac{64}{15} = \frac{128}{15}$.

8. Find the area of the shaded region (pictured)

Answer: If we integrate with respect to x, we will have to break the region into two pieces, integrate them separately, and then add. Instead, let's integrate with respect to y. Then the left curve is given by $x = \sqrt{y}$ and the right curve by $x = 2 - y$. The point of intersection of the curves is where

$$\sqrt{y} = 2 - y.$$

Hence, $y = (2 - y)^2 = 4 - 4y + y^2$, so $y^2 - 5y + 4 = 0$. This factors as

$$(y - 4)(y - 1) = 0,$$

so we see that the curves intersect where $y = 1$ and $y = 4$. From looking at the picture, it's clear that $y = 1$ is the intersection we're interested in. Since we always subtract the leftmost curve from the rightmost when we're integrating in term of y, the shaded area is given by

$$\begin{aligned} A &= \int_0^1 [(2-y) - \sqrt{y}]\, dy \\ &= 2y - \frac{y^2}{2} - \frac{2}{3}y^{3/2}\Big]_0^1 \\ &= (2 - \frac{1}{2} - \frac{2}{3}) - 0 \\ &= \frac{7}{6}. \end{aligned}$$

9. Find the total shaded area (pictured)

Answer: There are two separate pieces of the shaded region, so we deal with them separately. The two curves intersect where

$$x^2 - 4 = -x^2 - 2x,$$

which, when we add both terms on the right becomes $2x^2 + 2x - 4 = 0$. This factors as

$$(2x-2)(x+2) = 0,$$

so the points of intersection are at $x = -2$ and $x = 1$. Hence, the lefthand region corresponds to the interval $[-3, -2]$ and the righthand region to the interval $[-2, 1]$. Finally, note that $y = x^2 - 4$ is the top curve on the left region, and $y = -x^2 - 2x$ is the top curve on the right region, so the total area is given by

$$\begin{aligned} A &= \int_{-3}^{-2} \left[(x^2-4) - (-x^2-2x)\right] dx + \int_{-2}^{1} \left[(-x^2-2x) - (x^2-4)\right] dx \\ &= \int_{-3}^{-2} \left[2x^2 + 2x - 4\right] dx + \int_{-2}^{1} \left[-2x^2 - 2x + 4\right] dx \\ &= \left[\frac{2}{3}x^3 + x^2 - 4x\right]_{-3}^{-2} + \left[\frac{-2}{3}x^3 - x^2 + 4x\right]_{-2}^{1} \\ &= \frac{38}{3} \end{aligned}$$

27. Find the area of the region enclosed by the curves $x + y^2 = 0$ and $x + 3y^2 = 2$.

Answer: Note that we can write these curves as $x = -y^2$ and $x = 2 - 3y^2$. Hence, they intersect where

$$-y^2 = 2 - 3y^2.$$

Solving for y, we see that $2y^2 - 2 = 0$. Dividing by 2, $y^2 - 1 = 0$, so we see that $y = \pm 1$. Hence, the curves intersect at $y = 1$ and $y = -1$. If you've drawn the picture correctly (which I hope you've done), then it's clear that $x = 2 - 3y^2$ is the rightmost curve, so the area of the region we're interested in is given by

$$\begin{aligned} A &= \int_{-1}^{1} \left[(2-3y^2) - (-y^2)\right] dy \\ &= \int_{-1}^{1} \left[2-2y^2\right] \\ &= 2y - \frac{2}{3}y^3\Big]_{-1}^{1} \\ &= \left(2-\frac{2}{3}\right) - \left(-2+\frac{2}{3}\right) \\ &= \frac{8}{3}. \end{aligned}$$

31. Find the area of the region enclosed by the curves $4x^2 + y = 4$ and $x^4 - y = 1$.

Answer: Note that we can re-write these curves as $y = 4 - 4x^2$ and $y = x^4 - 1$, respectively. The curves intersect where

$$4 - 4x^2 = x^4 - 1,$$

which is to say, where $x^4 + 4x^2 - 5 = 0$. This factors as

$$(x^2+5)(x^2-1) = 0,$$

so $x^2 = -5$ or $x^2 = 1$. Clearly, we don't want x^2 to be negative, so $x^2 = 1$ and so the points of intersection are at $x = \pm 1$. Now, since $y = 4 - 4x^2$ is the topmost curve (draw a picture), the area is

$$\begin{aligned} A &= \int_{-1}^{1} \left[(4-4x^2) - (x^4-1)\right] dx \\ &= \int_{-1}^{1} \left[5 - 4x^2 - x^4\right] dx \\ &= 5x - \frac{4}{3}x^3 - \frac{x^5}{5}\Big]_{-1}^{1} \\ &= \left(5 - \frac{4}{3} - \frac{1}{5}\right) - \left(-5 + \frac{4}{3} + \frac{1}{5}\right) \\ &= \frac{104}{15}. \end{aligned}$$

§5.2

1. Find the formula for the area $A(x)$ of the cross sections of the following solids. In each case, the solid lies between planes perpindicular to the x-axis at $x = -1$ and $x = 1$. In each case, the cross sections perpindicular to the x-axis between these planes run from the semicircle $y = -\sqrt{1-x^2}$ to the semicircle $y = \sqrt{1-x^2}$.

(a) The cross sections are disks with diameters in the xy-plane

Answer: Since the diameter of each cross-sectional disk lies in the plane, the radius of each disk is simply y. Hence, $A(x) = \pi y^2 = \pi \sqrt{1-x^2}^2 = \pi(1-x^2)$.

(b) The cross sections are squares with bases in the xy-plane.

Answer: Since the base of each cross-section lies in the plane, that means that the length of a side of a cross-sectional square is simply $2y = 2\sqrt{1-x^2}$. Hence, the area of a square is simply $A(x) = (2y)^2 = (2\sqrt{1-x^2})^2 = 4-4x^2$.

(c) The cross sections are squares with diagonals in the xy-plane.

Answer: Since the length of a square's diagonal is $\sqrt{2}$ times the length of one of its sides, the length of a side is $\frac{1}{\sqrt{2}}$ times the length of the diagonal. Since each square's diagonal is just the diameter of the circle, which is to say $2y$, we see that the length of the side of a cross-sectional square is $\frac{1}{\sqrt{2}}(2y) = \sqrt{2}y$. Hence, the area of each square is given by

$$A(x) = (\sqrt{2}y)^2 = \left(\sqrt{2}\sqrt{1-x^2}\right)^2 = 2(1-x^2) = 2-2x^2.$$

(d) The cross sections are equilateral triangles with bases in the xy-plane.

Answer: Note that each cross-sectional triangle is half of one of the cross-sectional squares from part (c). Hence, the area of a cross-sectional triangle is half the area of one of the squares in part (c). Therefore, $A(x) = \frac{1}{2}(2-2x^2) = 1-x^2$.

10. Find the volume of the solid with base given by the disk $x^2 + y^2 \leq 1$. The cross sections by planes perpendicular to the y-axis between $y = -1$ and $y = 1$ are isosceles triangles with one leg in the disk.

Answer: Note that the base of each triangle is simply a diameter of the base circle. Hence, both the base and height of each cross-sectional triangle is the length of the corresponding diameter. Since our cross-sections are perpendicular to the y-axis, we must integrate with respect to y, so we want to express the areas of the triangles as a function of y. The equation of the circle can be re-written as

$$x = \pm\sqrt{1-y^2}.$$

Now, note that the radius of the circle at each point is given by x, so the diameter is given by $2x$. Since we know the diameter is the same length as both the base and height of the corresponding cross-sectional triangle and the area of a triangle is given by $\frac{1}{2}bh$, we see that

$$A(y) = \frac{1}{2}(2x)(2x) = 2x^2 = 2(\sqrt{1-y^2})^2 = 2(1-y^2) = 2-2y^2.$$

Therefore, the volume of the solid is given by

$$\begin{aligned} V &= \int_{-1}^{1} \left[2 - 2y^2\right] dy \\ &= 2y - \frac{2}{3}y^3\bigg]_{-1}^{1} \\ &= \left(2 - \frac{2}{3}\right) - \left(-2 + \frac{2}{3}\right) \\ &= \frac{8}{3}. \end{aligned}$$

DRL 3E3A, University of Pennsylvania
E-mail address: shonkwil@math.upenn.edu