

MATH 104 HW 3

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§5.3

2. Find the volume of the solid generated by revolving the region bounded by $x = 3y/2$, $y = 2$ around the y -axis.

Answer: First, note that $3y/2 = 2$ when $3y = 4$; i.e. when $y = 4/3$. Hence, the volume is given by

$$V = \int_0^{4/3} \pi r^2 dy$$

Now, the radius of the disk is simply $x = 3y/2$, so

$$\begin{aligned} V &= \int_0^{4/3} \pi (3y/2)^2 dy \\ &= \pi \int_0^{4/3} \frac{9y^2}{4} dy \\ &= \pi \left[\frac{1}{3} \frac{9y^3}{4} \right]_0^{4/3} \\ &= \frac{3\pi}{4} \left(\left(\frac{4}{3} \right)^3 - 0 \right) \\ &= \frac{16\pi}{9}. \end{aligned}$$

4. Done in class

6. Find the volume of the solid generated by revolving the region bounded by $y = x^3$, $y = 0$ and $x = 2$ about the x -axis.

Answer: The volume is given by

$$V = \int_0^2 \pi r^2 dx.$$

Now, the radius of the disk is simply given by $y = x^3$, so

$$\begin{aligned} V &= \int_0^2 \pi(x^3)^2 dx \\ &= \pi \int_0^2 x^6 dx \\ &= \pi \left[\frac{x^7}{7} \right]_0^2 \\ &= \pi \left(\frac{128}{7} - 0 \right) \\ &= \frac{128\pi}{7}. \end{aligned}$$

39. Find the volume of the solid generated by revolving the region bounded by $y = x^2$ and the line $y = 1$ about

(a): the line $y = 1$.

Answer: Note that $y = x^2$ and $y = 1$ intersect when $x = \pm 1$. Now, if we look at the picture, the radius is given by $1 - x^2$, so

$$\begin{aligned} V &= \int_{-1}^1 \pi r^2 dx \\ &= \int_{-1}^1 \pi(1 - x^2)^2 dx \\ &= \pi \int_{-1}^1 [1 - 2x^2 + x^4] dx \\ &= \pi \left[x - \frac{2}{3}x^3 + \frac{x^5}{5} \right]_{-1}^1 \\ &= \pi \left[\left(1 - \frac{2}{3} + \frac{1}{5} \right) - \left(-1 + \frac{2}{3} - \frac{1}{5} \right) \right] \\ &= \frac{16\pi}{15}. \end{aligned}$$

(b): The line $y = 2$.

Answer: This is like part (a), except now we have a hole in the middle of our solid. If you look at the picture, it's easy to see that the outer radius, R , of the solid is given by $2 - x^2$, whereas the inner

radius, r , is simply 1. Hence,

$$\begin{aligned}
 V &= \int_{-1}^1 \pi(R^2 - r^2) dx \\
 &= \int_{-1}^1 \pi \left((2 - x^2)^2 - 1^2 \right) dx \\
 &= \pi \int_{-1}^1 (3 - 4x^2 + x^4) dx \\
 &= \pi \left[3x - \frac{4}{3}x^3 + \frac{x^5}{5} \right]_{-1}^1 \\
 &= \frac{56\pi}{15}
 \end{aligned}$$

(c): The line $y = -1$.

Answer: As in (b), we now have a hole. The outer radius (look at the picture) is given by $R = x^2 - (-1) = x^2 + 1$, while the inner radius is given by $r = 1$. Hence,

$$\begin{aligned}
 V &= \int_{-1}^1 \pi(R^2 - r^2) dx \\
 &= \pi \int_{-1}^1 \left((x^2 + 1)^2 - 1^2 \right) dx \\
 &= \pi \int_{-1}^1 (x^4 + 2x^2) dx \\
 &= \pi \left[\frac{x^5}{5} + \frac{2}{3}x^3 \right]_{-1}^1 \\
 &= \frac{26\pi}{15}
 \end{aligned}$$

§5.4

1. Use the shell method to find the volume of the solid generated by revolving the region bounded by $x = 0$, $x = 2$, $y = 0$ and $y = 1 + \frac{x^2}{4}$ about the y -axis.

Answer: Recall that the shell method tells us that

$$V = \int_0^2 2\pi r h dx.$$

Now, in this case, r is just x and $h = y = 1 + \frac{x^2}{4}$. Hence,

$$\begin{aligned} V &= \int_0^2 2\pi x \left(1 + \frac{x^2}{4}\right) dx \\ &= 2\pi \int_0^2 \left(x + \frac{x^3}{4}\right) dx \\ &= 2\pi \left[\frac{x^2}{2} + \frac{x^4}{16}\right]_0^2 \\ &= 2\pi(2 + 1) - 0 \\ &= 6\pi. \end{aligned}$$

4. Find the volume of the solid generated by revolving the region bounded by $y = \sqrt{3}$, $x = 3$ and $x = 3 - y^2$ about the x -axis.

Answer: Since we're revolving about the x -axis, we need to integrate with respect to y (using the shell method). Hence, remembering that in such a circumstance h represents the *width* of the region,

$$\begin{aligned} V &= \int_0^{\sqrt{3}} 2\pi r h dy \\ &= \int_0^{\sqrt{3}} 2\pi y(3 - (3 - y^2)) dy \\ &= 2\pi \int_0^{\sqrt{3}} y^3 dy \\ &= 2\pi \left[\frac{y^4}{4}\right]_0^{\sqrt{3}} \\ &= \frac{9\pi}{2} \end{aligned}$$

13. Find the volume of the solid generated by revolving the region bounded by $y = 1/x$, $y = 0$, $x = 1/2$ and $x = 2$ about the y -axis.

Answer: Since we're revolving around the y -axis, we can integrate in terms of x . Hence,

$$\begin{aligned} V &= \int_{1/2}^2 2\pi r h dx \\ &= \int_{1/2}^2 2\pi x(1/x) dx \\ &= 2\pi \int_{1/2}^2 dx \\ &= 2\pi x \Big|_{1/2}^2 \\ &= 3\pi. \end{aligned}$$

15. Find the volume of the solid generated by revolving the region bounded by $x = \sqrt{y}$, $x = -y$ and $y = 2$ about the x -axis.

Answer: Since we're revolving around the x -axis, we must integrate in terms of y . Hence, $r = y$ and $h = \sqrt{y} - (-y) = \sqrt{y} + y$, so

$$\begin{aligned} V &= \int_0^2 2\pi r h dy \\ &= \int_0^2 2\pi y (\sqrt{y} + y) dy \\ &= 2\pi \int_0^2 (y^{3/2} + y^2) dy \\ &= 2\pi \left[\frac{2}{5} y^{5/2} + \frac{y^3}{3} \right]_0^2 \\ &= 2\pi \left[\left(\frac{2}{5} 4\sqrt{2} + \frac{8}{3} \right) - 0 \right] \\ &= \pi \left[\frac{16\sqrt{2}}{5} + \frac{16}{3} \right]. \end{aligned}$$

23. Find the volume of the solid generated by revolving the region bounded by $x = 12(y^2 - y^3)$ and $x = 0$ about:

(a): The x -axis.

Answer: In this case, $r = y$ and $h = x = 12(y^2 - y^3)$, so

$$\begin{aligned} V &= \int_0^1 2\pi r h dy \\ &= \int_0^1 2\pi y (12(y^2 - y^3)) dy \\ &= 24\pi \int_0^1 (y^3 - y^4) dy \\ &= 24\pi \left[\frac{y^4}{4} - \frac{y^5}{5} \right]_0^1 \\ &= 24\pi \left[\left(\frac{1}{4} - \frac{1}{5} \right) - 0 \right] \\ &= \frac{24\pi}{20} \\ &= \frac{6\pi}{5} \end{aligned}$$

(b): The line $y = 1$.

Answer: In this case, $r = 1 - y$ and h is the same as in (a), so

$$\begin{aligned}
 V &= \int_0^1 2\pi r h dy \\
 &= \int_0^1 2\pi(1-y)(12(y^2 - y^3)) dy \\
 &= 24\pi \int_0^1 (y^2 - 2y^3 + y^4) dy \\
 &= 24\pi \left[\frac{y^3}{3} - \frac{2y^4}{4} + \frac{y^5}{5} \right]_0^1 \\
 &= 24\pi \left[\left(\frac{1}{3} - \frac{1}{2} + \frac{1}{5} \right) - 0 \right] \\
 &= \frac{24\pi}{30} \\
 &= \frac{4\pi}{5}
 \end{aligned}$$

(c): The line $y = 8/5$.

Answer: Now, $r = 8/5 - y$ and h is still as in (a) and (b), so

$$\begin{aligned}
 V &= \int_0^1 2\pi r h dy \\
 &= \int_0^1 2\pi(8/5 - y)(12(y^2 - y^3)) dy \\
 &= 24\pi \int_0^1 \left(\frac{8y^2}{5} - \frac{13y^3}{5} + y^4 \right) dy \\
 &= 24\pi \left[\frac{8y^3}{15} - \frac{13y^4}{20} + \frac{y^5}{5} \right]_0^1 \\
 &= 24\pi \left[\left(\frac{8}{15} - \frac{13}{20} + \frac{1}{5} \right) - 0 \right] \\
 &= 2\pi.
 \end{aligned}$$

(d): The line $y = -2/5$.

Answer: Now, $r = y - (-2/5) = y + 2/5$ and h is as above, so

$$\begin{aligned} V &= \int_0^1 2\pi r h dy \\ &= \int_0^1 2\pi(y + 2/5)(12(y^2 - y^3)) dy \\ &= 24\pi \int_0^1 \left(\frac{2y^2}{5} + \frac{3y^3}{5} - y^4 \right) dy \\ &= 24\pi \left[\frac{2y^3}{15} + \frac{3y^4}{20} - \frac{y^5}{5} \right]_0^1 \\ &= 24\pi \left[\left(\frac{2}{15} + \frac{3}{20} - \frac{1}{5} \right) - 0 \right] \\ &= 2\pi. \end{aligned}$$

38. The region shown here (bounded by $y = x^2$, $y = -x^4$ and $x = 1$) is to be revolved about the y -axis to generate a solid. Which of the methods (disk, washer, shell) could you use to find the volume of the solid? How many integrals would be required in each case? Give reasons for your answers.

Answer: Both the washer and shell methods will find the volume of the solid. However, the shell method finds it with a single integral, namely

$$V = \int_0^1 2\pi x(x^2 - (-x^4)) dx = \int_0^1 2\pi x(x^2 + x^4) dx.$$

On the other hand, to find the volume using the washer method, we must use two separate integrals, one for the part of the region above the x -axis, and the other for the part below the x -axis. Using the washer method, the volume is given by

$$V = \int_{-1}^0 \pi(1^2 - (-x^4)^2) dy + \int_0^1 \pi(1^2 - (x^2)^2) dy.$$

§5.5

13. Done in class

18. Find the length of the curve

$$y = \int_{-2}^x \sqrt{3t^4 - 1} dt$$

on the region $-2 \leq x \leq -1$.

Answer: We want to use the length-of-curve formula

$$L = \int_{-2}^{-1} \sqrt{1 + \left(\frac{dy}{dx} \right)^2} dx.$$

To do so, we note that, by the fundamental theorem of calculus,

$$\frac{dy}{dx} = \frac{d}{dx} \int_{-2}^x \sqrt{3t^4 - 1} dt = \sqrt{3x^4 - 1},$$

so

$$\left(\frac{dy}{dx}\right)^2 = 3x^4 - 1.$$

Hence,

$$\begin{aligned} L &= \int_{-2}^{-1} \sqrt{1 + (3x^4 - 1)} dx \\ &= \int_{-2}^{-1} \sqrt{3x^4} dx \\ &= \int_{-2}^{-1} \sqrt{3} x^2 dx \\ &= \left. \sqrt{3} \frac{x^3}{3} \right]_{-2}^{-1} \\ &= \frac{\sqrt{3}}{3} [(-1)^3 - (-2)^3] \\ &= \frac{7\sqrt{3}}{3}. \end{aligned}$$

22. Find the length of the astroid given by $x^{2/3} + y^{2/3} = 1$ by finding the length of half the first-quadrant portion, $y = (1 - x^{2/3})^{3/2}$, $\sqrt{2}/4 \leq x \leq 1$, and multiplying by 8.

Answer: Since $y = (1 - x^{2/3})^{3/2}$, we know, by the chain rule, that

$$\frac{dy}{dx} = \frac{3}{2}(1 - x^{2/3})^{1/2} \cdot \left(-\frac{2}{3}x^{-1/3}\right) = -\frac{\sqrt{1 - x^{2/3}}}{\sqrt[3]{x}}.$$

Hence,

$$\left(\frac{dy}{dx}\right)^2 = \left(-\frac{\sqrt{1 - x^{2/3}}}{\sqrt[3]{x}}\right)^2 = \frac{1 - x^{2/3}}{x^{2/3}}.$$

Therefore,

$$1 + \left(\frac{dy}{dx}\right)^2 = 1 + \frac{1 - x^{2/3}}{x^{2/3}} = \frac{x^{2/3}}{x^{2/3}} + \frac{1 - x^{2/3}}{x^{2/3}} = \frac{1}{x^{2/3}}.$$

Thus, by the length-of-curve formula,

$$\begin{aligned} L &= \int_{\sqrt{2}/4}^1 \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx \\ &= \int_{\sqrt{2}/4}^1 \sqrt{\frac{1}{x^{2/3}}} dx \\ &= \int_{\sqrt{2}/4}^1 \frac{1}{x^{1/3}} dx \\ &= \int_{\sqrt{2}/4}^1 x^{-1/3} dx \\ &= \left. \frac{3}{2} x^{2/3} \right]_{\sqrt{2}/4}^1 \\ &= \frac{3}{2} - \frac{3}{2} \left(\frac{\sqrt{2}}{4}\right)^{2/3} \\ &= \frac{3}{2} - \frac{3\sqrt[3]{2}}{4}. \end{aligned}$$

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