MATH 114

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Problem 40, §11.1

A particle traveling in a straight line is located at the point $(1, -1, 2)$ and has speed 2 at time $t = 0$. The particle moves toward the point $(3, 0, 3)$ with constant acceleration $2\mathbf{i} + \mathbf{j} + \mathbf{k}$. Find its position vector $\mathbf{r}(t)$ at time t.

Solution: We're given that $\mathbf{a} = 2\mathbf{i} + \mathbf{j} + \mathbf{k}$. We also know that velocity and acceleration are related to position; namely,

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} \text{ and } \mathbf{a} = \frac{d\mathbf{v}}{dt}.$$

Hence,

$$\begin{aligned} \mathbf{v}(t) = \int \mathbf{a}(t)dt &= \int [2\mathbf{i} + \mathbf{j} + \mathbf{k}]\, dt \\ &= (2t)\mathbf{i} + (t)\mathbf{j} + (t)\mathbf{k} + \mathbf{c} \qquad (1) \\ &= (2t + c_1)\mathbf{i} + (t + c_2)\mathbf{k} + (t + c_3)\mathbf{k}, \end{aligned}$$

where $\mathbf{c} = c_1\mathbf{i} + c_2\mathbf{j} + c_3\mathbf{k}$ is a constant vector. Note that, therefore, $\mathbf{v}(0) = c_1\mathbf{i} + c_2\mathbf{j} + c_3\mathbf{k}$.

Now, the vector from $(1, -1, 2)$ to $(3, 0, 3)$ is given by:

$$(3 - 1)\mathbf{i} + (0 - (-1))\mathbf{j} + (3 - 2)\mathbf{k} = 2\mathbf{i} + \mathbf{j} + \mathbf{k}.$$

Since the particle is traveling along the straight line from $(1, -1, 2)$ to $(3, 0, 3)$, it is moving in the direction of the vector $2\mathbf{i} + \mathbf{j} + \mathbf{k}$; furthermore, since the velocity is always tangent to the direction of motion, we know that the velocity vector is also pointed in this same direction. Hence, $\mathbf{v}(0)$ must be some positive scalar multiple of $2\mathbf{i} + \mathbf{j} + \mathbf{k}$, so there is some scalar A such that

$$c_1\mathbf{i} + c_2\mathbf{j} + c_3\mathbf{k} = \mathbf{v}(0) = A(2\mathbf{i} + \mathbf{j} + \mathbf{k}). \qquad (2)$$

However, this isn't all we know about $\mathbf{v}$; we also know that the particle's speed at time $t = 0$ is 2, which is to say that $|\mathbf{v}(0)| = 2$. Hence, using the equality (2),

$$2 = |\mathbf{v}(0)| = |A(2\mathbf{i} + \mathbf{j} + \mathbf{k})| = |A||2\mathbf{i} + \mathbf{j} + \mathbf{k}| = A\sqrt{2^2 + 1^2 + 1^2} = A\sqrt{6},$$

so $A = \frac{2}{\sqrt{6}}$. Again, referring to (2), this means that

$$c_1\mathbf{i} + c_2\mathbf{j} + c_3\mathbf{k} = A(2\mathbf{i} + \mathbf{j} + \mathbf{k}) = \frac{2}{\sqrt{6}}(2\mathbf{i} + \mathbf{j} + \mathbf{k}) = \frac{4}{\sqrt{6}}\mathbf{i} + \frac{2}{\sqrt{6}}\mathbf{j} + \frac{2}{\sqrt{6}}\mathbf{k}.$$

Reading off the values of c_1, c_2, c_3, and incorporating them into the formula for $\mathbf{v}$ given in (1), we see that

$$\mathbf{v}(t) = \left(2t + \frac{4}{\sqrt{6}}\right)\mathbf{i} + \left(t + \frac{2}{\sqrt{6}}\right)\mathbf{j} + \left(t + \frac{2}{\sqrt{6}}\right)\mathbf{k}.$$

Since $\mathbf{v} = \frac{d\mathbf{r}}{dt}$, we know that

$$\begin{aligned}\mathbf{r}(t) = \int \mathbf{v}(t)dt &= \int \left[\left(2t + \frac{4}{\sqrt{6}}\right)\mathbf{i} + \left(t + \frac{2}{\sqrt{6}}\right)\mathbf{j} + \left(t + \frac{2}{\sqrt{6}}\right)\mathbf{k}\right] dt \\ &= \left(t^2 + \frac{4}{\sqrt{6}}\right)\mathbf{i} + \left(\frac{t^2}{2} + \frac{2}{\sqrt{6}}\right)\mathbf{j} + \left(\frac{t^2}{2} + \frac{2}{\sqrt{6}}\right)\mathbf{k} + \mathbf{c}' \qquad (3)\end{aligned}$$

where $\mathbf{c}' = c_4\mathbf{i} + c_5\mathbf{j} + c_6\mathbf{k}$ is a constant vector. Now, at $t = 0$, we know that $\mathbf{r}(t) = (1, -1, 2)$. Hence,

$$\mathbf{i} - \mathbf{j} + 2\mathbf{k} = \mathbf{r}(0) = \mathbf{c}'.$$

Therefore, plugging this into (3), we see that

$$\mathbf{r}(t) = \left(t^2 + \frac{4}{\sqrt{6}} + 1\right)\mathbf{i} + \left(\frac{t^2}{2} + \frac{2}{\sqrt{6}} - 1\right)\mathbf{j} + \left(\frac{t^2}{2} + \frac{2}{\sqrt{6}} + 2\right)\mathbf{k}.$$

This looks pretty nasty, so notice that we can factor this into

$$\mathbf{r}(t) = \left(\frac{1}{2}t^2 + \frac{2}{\sqrt{6}}t\right)(2\mathbf{i} + \mathbf{j} + \mathbf{k}) + (\mathbf{i} - \mathbf{j} + 2\mathbf{k}).$$

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