

## TOPOLOGY HW 2

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### 18.1

Prove that for functions  $f : \mathbb{R} \rightarrow \mathbb{R}$ , the  $\epsilon$ - $\delta$  definition of continuity implies the open set definition.

*Proof.* Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be continuous under the  $\epsilon$ - $\delta$  definition of continuity. Then we want to show that  $f$  is continuous under the open set definition. Let  $(a, b)$  be a basis element of the standard topology of  $\mathbb{R}$ . Let  $x \in f^{-1}(a, b)$ . Then  $f(x) \in (a, b)$ . Let  $\epsilon = \min\{f(x) - a, b - f(x)\}$ . Then, since  $f$  is continuous under the  $\epsilon$ - $\delta$  definition, there exists  $\delta > 0$  such that, if  $|x - y| < \delta$ ,

$$|f(x) - f(y)| < \epsilon$$

which is to say

$$f(x - \delta, x + \delta) \subseteq (f(x) - \epsilon, f(x) + \epsilon) \subseteq (a, b).$$

In turn, this means that

$$(x - \delta, x + \delta) \subseteq f^{-1}(a, b).$$

$(x - \delta, x + \delta)$  is open and contains  $x$ , so  $f^{-1}(a, b)$  is open, meaning  $f$  is continuous under the open set definition. Since our choice of  $f$  was arbitrary, we can conclude that the  $\epsilon$ - $\delta$  definition of continuity implies the open set definition. □

### 18.6

Find a function  $f : \mathbb{R} \rightarrow \mathbb{R}$  that is continuous at precisely one point.

**Example:** Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be defined by

$$f(x) = \begin{cases} x^2 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}$$

Since we've just shown that the  $\epsilon$ - $\delta$  definition of continuity is equivalent to the open set definition, we use an  $\epsilon$ - $\delta$  argument to show that  $f$  is continuous at 0 and nowhere else. Let  $\epsilon > 0$ . Define  $\delta < \sqrt{\epsilon}$ . Then, if  $|x - 0| < \delta$ ,

$$|f(x) - f(0)| = |f(x)| \leq |x^2| = |x|^2 < |\delta|^2 = \epsilon.$$

Hence,  $f$  is continuous at 0. Suppose  $f$  is continuous at some point  $c \neq 0$ . Then, for  $\epsilon < c^2/2$ , there exists  $\delta > 0$  such that, if  $|x - c| < \delta$ ,

$$|f(x) - f(c)| < \epsilon.$$

If  $c$  is rational, then there exists an irrational number  $b \in (c - \delta, c + \delta)$ . Then

$$|f(b) - f(c)| = |0 - c^2| = c^2 > \epsilon.$$

On the other hand, suppose  $c$  is irrational. Since  $f(x) = x^2$  is continuous, there exists  $\delta_0 > 0$  such that

$$|x - c| < \delta_0 \Rightarrow |f(x) - f(c)| < \epsilon.$$

Let  $\delta_1 = \min\{\delta, \delta_0\}$ . Then there exists rational  $a \in (c - \delta_1, c + \delta_1)$ , so

$$|f(a) - f(c)| = |a^2 - 0| = a^2$$

where

$$|c^2 - a^2| < \epsilon$$

or  $a^2 > \epsilon$ . In either case, we see that  $f$  is not continuous at any point other than 0.



## 18.8

Let  $Y$  be an ordered set in the order topology. Let  $f, g : X \rightarrow Y$  be continuous.

(a) Show that the set  $\{x | f(x) \leq g(x)\}$  is closed in  $X$ .

**Lemma 0.1.** *If  $Z$  is a topological space under the order topology, then  $Z$  is Hausdorff.*

*Proof.* Let  $Z$  be a topological space under the order topology. Let  $x, y \in Y$  such that  $x < y$ . If there exists  $z \in Z$  such that  $x < z < y$ , then we can construct disjoint open sets containing  $x$  and  $y$ , respectively, in the following way:

If there exist  $a, b \in Z$  such that  $a < x$  and  $y < b$ , then  $(a, z)$  and  $(z, b)$  are open in  $Z$ , are disjoint, and contain  $x$  and  $y$ , respectively. If  $x$  is the minimal element in  $Z$ , then, if  $b > y$ ,  $[x, z), (z, b)$  are disjoint, open in  $Z$  and contain  $x$  and  $y$ , respectively. If  $y$  is the maximal element of  $Z$ , then, if there exists  $a < x$ ,  $(a, z), (z, y]$  are disjoint, open in  $Z$  and contain  $x$  and  $y$ , respectively. Finally, if  $x$  is the minimal element and  $y$  is the maximal element in  $Z$ , then  $[x, z), (z, y]$  are disjoint, open in  $Z$  and contain  $x$  and  $y$ , respectively. Hence, whenever there exists  $z \in Z$  such that  $x < z < y$ , we can construct disjoint open sets containing  $x$  and  $y$  respectively.

On the other hand, if there exists no  $z \in Z$  such that  $x < z < y$ , then, if  $a < x$  and  $b > y$ ,  $(a, y), (x, b)$  are disjoint open sets in  $Z$  that contain  $x$  and  $y$ , respectively. Again, if  $x$  is the minimal element and/or  $y$  is the maximal element in  $Z$ , we can still construct two disjoint open sets, one containing  $x$  and one containing  $y$ .

Hence, we can conclude that  $Z$  is Hausdorff. □

**Proposition 0.2.**  $\{x | f(x) \leq g(x)\}$  is closed in  $X$ .

*Proof.* To show that  $\{x|f(x) \leq g(x)\}$  is closed, it is sufficient to demonstrate that  $X \setminus \{x|f(x) \leq g(x)\} = \{x|f(x) > g(x)\} = A$  is open in  $X$ . Let  $x_0 \in A$ . Then  $g(x_0) < f(x_0)$ . Since  $Y$  is Hausdorff by the above lemma, there exist disjoint open sets  $U$  and  $V$  contained in  $Y$  such that  $f(x_0) \in U$ ,  $g(x_0) \in V$ . Then, since  $f, g$  are continuous,  $f^{-1}(U)$  and  $g^{-1}(V)$  are open in  $X$ , so their intersection

$$f^{-1}(U) \cap g^{-1}(V)$$

is open in  $X$ . Furthermore,  $x_0 \in f^{-1}(U) \cap g^{-1}(V)$ , so there exists open  $W \subseteq f^{-1}(U) \cap g^{-1}(V)$  containing  $x_0$ . Since  $U$  and  $V$  are disjoint,

$$f(W) \cap g(W) = \emptyset$$

meaning, for all  $w \in W$ ,

$$g(w) < f(w).$$

In other words,  $W \subseteq A$ . Therefore, we can conclude that  $A$  is open, meaning that its complement,  $\{x|f(x) \leq g(x)\}$  is closed.  $\square$

(b) Let  $h : X \rightarrow Y$  be the function

$$h(x) \min\{f(x), g(x)\}.$$

Show that  $h$  is continuous.

*Proof.* By a similar argument to that made in (a) above, we can show that  $B = \{x|g(x) \leq f(x)\}$  is closed. Also, since  $f$  and  $g$  are continuous on  $X$ , it is true that  $f$  is continuous on  $A = \{x|f(x) \leq g(x)\}$  and  $g$  is continuous on  $B$ . Furthermore,  $A \cup B = X$  and  $A \cap B = \{x|f(x) = g(x)\}$ , which is to say that  $f(x) = g(x)$  for all  $x \in A \cap B$ . By the Pasting Lemma, then, we can construct  $h' : X \rightarrow Y$  such that  $h'(x) = f(x)$  when  $x \in A$  and  $h'(x) = g(x)$  when  $x \in B$ . Finally, we need only note that  $h \equiv h'$  to demonstrate that  $h$  is continuous.  $\square$

### 18.9

Let  $\{A_\alpha\}$  be a collection of subsets of  $X$ ; let  $X = \cup_\alpha A_\alpha$ . Let  $f : X \rightarrow Y$ ; suppose that  $f|_{A_\alpha}$  is continuous for each  $\alpha$ .

(a) Show that if the collection  $\{A_\alpha\}$  is finite and each set  $A_\alpha$  is closed, then  $f$  is continuous.

*Proof.* If the collection  $\{A_\alpha\}$  is finite, then  $X = \cup_{i=1}^n A_i$  for some integer  $n$ . If  $n = 2$ , then  $X = A_1 \cup A_2$ ,  $A_1$  and  $A_2$  are closed in  $X$ ,  $f|_{A_1} : A_1 \rightarrow Y$  and  $f|_{A_2} : A_2 \rightarrow Y$  are continuous and  $f|_{A_1}(x) = f|_{A_2}(x)$  for every  $x \in A_1 \cap A_2$ . Hence, by the pasting lemma, we can construct continuous  $f' : X \rightarrow Y$  such that  $f'(x) = f|_{A_1}(x)$  if  $x \in A_1$  and  $f'(x) = f|_{A_2}(x)$  if  $x \in A_2$ . It is clear that  $f \equiv f'$ , so  $f$  is continuous.

Now, suppose that every map  $f$  fulfilling the above hypotheses is continuous on any  $X = \bigcup_{i=1}^n A_i$ . Let  $X = \bigcup_{i=1}^{n+1} A_i$ . Then,

$$\left( \bigcup_{i=1}^n A_i \right) \cup A_{n+1}.$$

Since  $\bigcup_{i=1}^n A_i$  is a finite union of closed sets, it is closed, as is  $A_{n+1}$ . Furthermore, by the inductive hypothesis,  $f|_{\bigcup_{i=1}^n A_i}$  is continuous, as is  $f|_{A_{n+1}}$ . Hence, by the pasting lemma, we can construct continuous  $f' : X \rightarrow Y$  such that  $f'(x) = f|_{\bigcup_{i=1}^n A_i}(x)$  if  $x \in \bigcup_{i=1}^n A_i$  and  $f'(x) = f|_{A_{n+1}}(x)$  if  $x \in A_{n+1}$ . Again,  $f \equiv f'$ , so  $f$  is continuous.

Therefore, by induction, if the collection  $\{A_\alpha\}$  is finite and each set  $A_\alpha$  is closed, then  $f$  is continuous.  $\square$

(b) Find an example where the collection  $\{A_\alpha\}$  is countable and each  $A_\alpha$  is closed, but  $f$  is not continuous.

**Example:** Let  $f : \mathbb{R}_{std} \rightarrow \mathbb{R}_K$  be the identity map. Then  $f$  is not continuous, as  $(-1, 1) - K$  is open in  $\mathbb{R}_K$  but not in  $\mathbb{R}$ , since no neighborhood of  $\{0\}$  is contained in  $(-1, 1) - K$ . Let

$$A_n = (-\infty, -\frac{1}{n}] \cup \{0\} \cup [\frac{1}{n}, \infty).$$

Then  $\{A_n\}$  is countable and  $\mathbb{R} = \bigcup_1^\infty A_n$ .

We want to show, then, that  $f|_{A_n} : A_n \rightarrow \mathbb{R}_K$  is continuous. Let  $U \subseteq \mathbb{R}_K$  be open. Then

$$\begin{aligned} f^{-1}|_{A_n}(U) &= f^{-1}|_{(-\infty, \frac{1}{n}]}(U) \cup f^{-1}|_{\{0\}}(U) \cup f^{-1}|_{[\frac{1}{n}, \infty)}(U) \\ &= (U \cap (-\infty, -\frac{1}{n}]) \cup (U \cap \{0\}) \cup (U \cap [\frac{1}{n}, \infty)) \end{aligned}$$

Since  $f$  is the identity map. Each of the three terms in this union are open in  $A_n$ , the outer two by definition of the subspace topology and  $U \cap \{0\}$  because

$$U \cap \{0\} = \begin{cases} \emptyset, & \text{open by definition} \\ \{0\} = A_n \cap (-\frac{1}{n+1}, \frac{1}{n+1}) \end{cases}$$

Which is open in  $A_n$ . Hence,  $f^{-1}|_{A_n}(U)$  is the union of three open sets in  $A_n$ , so  $f^{-1}|_{A_n}(U)$  is open. In turn, this implies that  $f|_{A_n}$  is continuous for all  $n$ .



(c) An indexed family of sets  $\{A_\alpha\}$  is said to be *locally finite* if each point  $x$  of  $X$  has a neighborhood that intersects  $A_\alpha$  for only finitely many values of  $\alpha$ . Show that if the family  $\{A_\alpha\}$  is locally finite and each  $A_\alpha$  is closed, then  $f$  is continuous.

*Proof.* Let  $\{A_\alpha\}$  be locally finite. Let  $x \in X$ . Then there exists a neighborhood  $U_x$  such that  $U_x \cap A_\alpha \neq \emptyset$  for only finitely many  $\alpha$ . Index the  $\alpha$  for

which this intersection is nonempty by  $\{\alpha_i\}_{i=1,\dots,n}$ . Then

$$f|_{U_x}(U_x) = \bigcup_{i=1}^n f|_{A_{\alpha_i}}(U_x).$$

Since each  $f|_{A_{\alpha_i}}$  is continuous, we can use an analog of the argument used in (a) above to show that the pasting lemma implies that  $f|_{U_x}$  is continuous. Since

$$X = \bigcup_{x \in X} U_x$$

then, by Theorem 18.2(f) (the local formulation of continuity),  $f$  is continuous.  $\square$

19.1

Prove Theorem 19.2

*Proof.* Let each  $X_\alpha$  be a space given by a basis  $\mathcal{B}_\alpha$ . Then we want to show that the collection of sets of the form

$$\mathcal{P} = \prod_{\alpha \in J} B_\alpha$$

is a basis for the box topology on  $\prod_{\alpha \in J} X_\alpha$ . Let

$$U = \prod_{\alpha \in J} U_\alpha$$

be an open set in the box topology and let  $x \in U$ . Then

$$x = (x_\alpha)_{\alpha \in J}$$

where each  $x_\alpha \in X_\alpha$ . For each  $x_\beta$ , there exists an element  $B_\beta \subseteq U_\beta$  of the basis for the topology on  $X_\beta$  that contains  $x_\beta$ . Since

$$\prod_{\alpha \in J} B_\alpha \in \mathcal{P},$$

we can conclude that, if  $\mathcal{T}'$  is a topology generated by the collection  $\mathcal{P}$ , then  $\mathcal{T}'$  will be finer than the box topology. Obviously, the box topology is finer than  $\mathcal{T}'$ , if it is a topology, as every basis element of  $\mathcal{T}'$  (again, assuming it is a topology) is contained in the standard basis for the box topology. Hence the equality of the topologies will be clear once we demonstrate that  $\mathcal{P}$  is, in fact, the basis of a topology. The fact that any element  $x \in \prod X_\alpha$  is contained in an element of  $\mathcal{P}$  is merely a special case of the result proved above, so we turn our attention to the second prerequisite for a basis. If  $x$  is contained in the intersection of two elements

$$x \in \left( \prod_{\alpha_a \in J} B_{\alpha_a} \right) \cap \left( \prod_{\alpha_b \in J} B_{\alpha_b} \right)$$

then, for each  $B_{\beta_a}, B_{\beta_b} \subseteq X_\beta$ ,  $x_\beta \in B_{\beta_a}$  and  $x_\beta \in B_{\beta_b}$ , which is to say that

$$x_\beta \in B_{\beta_a} \cap B_{\beta_b}.$$

Since  $X_\beta$  is a subspace, there exists a basis element

$$B_{\beta_c} \subseteq B_{\beta_a} \cap B_{\beta_b}$$

that contains  $x_\alpha$ . Then

$$\prod_{\alpha \in J} B_{\alpha_c} \subseteq \left( \prod_{\alpha \in J} B_{\alpha_a} \right) \cap \left( \prod_{\alpha \in J} B_{\alpha_b} \right)$$

is a basis element and contains  $x$ . Hence,  $\mathcal{B}_\alpha$  is a basis for a topology on  $\prod_{\alpha \in J} X_\alpha$ . Thus, our collection of  $\prod B_\alpha$  is a basis for the box topology.

Now, we turn our attention to the collection of all sets of the same form, where  $B_\alpha \in \mathcal{B}_\alpha$  for finitely many  $\alpha$  and  $B_\alpha = X_\alpha$  for all the remaining indices. We want to show that this collection  $\mathcal{Q}$  serves as a basis for the product topology on  $\prod_{\alpha \in J} X_\alpha$ . Let

$$U = \prod_{\alpha \in J} U_\alpha$$

be a basis element of the product topology and suppose  $x \in U$ . Let  $x = (x_\alpha) \in U$ . Then, for each  $\alpha$  for which  $U_\alpha \neq X_\alpha$  we assign an index  $U_{\alpha_i}$ . Then, since  $\mathcal{B}_{\alpha_j}$  is a basis for  $X_{\alpha_j}$ , there exists a basis element  $B_{\alpha_i}$  containing  $x_\alpha$  and contained in  $U_{\alpha_j}$ . Then the product

$$\prod_{\alpha \in J} B_\alpha$$

where  $B_\alpha = B_{\alpha_i}$  for each of the finite number of properly contained  $B_{\alpha_i}$  and  $B_\alpha = X_\alpha$  for all the remaining indices is an element of our collection. Thus, the topology  $\mathcal{T}$  generated by this basis (assuming it is a basis) is finer than the product topology on  $\prod X_\alpha$ . Furthermore, by arguments virtually identical to those above concerning the box topology,  $\mathcal{Q}$  is a basis for a topology on  $\prod X_\alpha$ . Finally, we note that any element of  $\mathcal{Q}$  is an element of the standard basis for the product topology, so  $\mathcal{Q}$  generates the product topology.  $\square$

## 19.2

Prove Theorem 19.3

*Proof.* Let

$$U = \prod_{\alpha \in J} U_\alpha$$

be a basis element as described in problem 19.1 above for either the box or product topology. Then, for each  $\alpha$ ,

$$U_\alpha \cap A_\alpha$$

is an element of the basis for the subspace topology on  $A_\alpha$ . Then it is clear that

$$B = \prod_{\alpha \in J} (U_\alpha \cap A_\alpha)$$

is an element of the basis of the box or product topology on  $\prod A_\alpha$ . Let  $x = (x_\alpha) \in B$ . Then each  $x_\alpha \in U_\alpha \cap A_\alpha$ . Specifically,  $x \in U$  and  $x \in \prod A_\alpha$ , so

$$x \in U \cap \prod_{\alpha \in J} A_\alpha$$

a basis element for the subspace topology on  $\prod A_\alpha$ . Hence the subspace topology on  $\prod A_\alpha$  is finer than the box or product topology on  $\prod A_\alpha$ .

On the other hand, if

$$x \in U \cap \prod_{\alpha \in J} A_\alpha,$$

an element of the basis for the subspace topology, then  $x_\alpha \in U_\alpha$  and  $x_\alpha \in A_\alpha$ . In other words,

$$x_\alpha \in U_\alpha \cap A_\alpha$$

for all  $\alpha$ , so

$$x \in \prod_{\alpha \in J} (U_\alpha \cap A_\alpha),$$

a basis element of the box or product topology on  $\prod A_\alpha$ . Therefore, the box or product topology on  $\prod A_\alpha$  is finer than the subspace topology.

Since each is finer than the other, we conclude that  $\prod A_\alpha$  is a subspace of  $\prod X_\alpha$  if both products are given either the box or the product topology.  $\square$

### 19.9

Show that the choice axiom is equivalent to the statement that for any indexed family  $\{A_\alpha\}_{\alpha \in J}$  of nonempty sets, with  $J \neq \emptyset$ , the cartesian product

$$\prod_{\alpha \in J} A_\alpha$$

is not empty.

*Proof.* According to the choice axiom, there exists a choice function

$$c : \{A_\alpha\}_{\alpha \in J} \rightarrow \bigcup_{\alpha \in J} A_\alpha$$

such that  $c(A_\alpha) = a_\alpha \in A_\alpha$ . Then

$$(c(A_\alpha))_{\alpha \in J} = (a_\alpha)_{\alpha \in J} \in \prod_{\alpha \in J} A_\alpha.$$

$\square$

## 19.10

Let  $A$  be a set; let  $\{X_\alpha\}_{\alpha \in J}$  be an indexed family of spaces; and let  $\{f_\alpha\}_{\alpha \in J}$  be an indexed family of functions  $f_\alpha : A \rightarrow X_\alpha$ .

(a) Show there is a unique coarsest topology  $\mathcal{T}$  on  $A$  relative to which each of the functions  $f_\alpha$  is continuous.

*Proof.* In order for  $f_\alpha$  to be continuous in  $\mathcal{T}$ , then, for each open  $U_\alpha \in X_\alpha$ ,  $f_\alpha^{-1}(U_\alpha)$  must be open in  $A$ . Furthermore, for any  $\alpha$ ,

$$f_\alpha^{-1}(X_\alpha) = A$$

so, if  $x \in A$ ,  $x \in f_\alpha^{-1}(X_\alpha)$ . So we let  $\mathcal{T}$  consist of all countable unions of finite intersections of such  $f_\alpha^{-1}(U_\alpha)$ , for all open  $U_\alpha$  in each  $X_\alpha$ . Suppose there exists a topology  $\mathcal{T}_1$  in which each  $f_\alpha$  is continuous which is coarser than  $\mathcal{T}$ . Then there exists an element  $U$  of  $\mathcal{T}$  which is not an element of  $\mathcal{T}_1$ . By construction,

$$U = \bigcup_{\alpha \in J} \left( \bigcap_{i=1}^n f_{\alpha_i}^{-1}(U_{\alpha_i}) \right)$$

where  $J$  is a countable set and each  $U_{\alpha_i}$  is an open set in some  $X_{\alpha_i}$ . Then, for each  $\alpha_i$ ,  $i = 1, \dots, n$ ,  $f_{\alpha_i}^{-1}(U_{\alpha_i})$  must be open in  $\mathcal{T}_1$ . Hence, since  $U$  is a countable union of a finite intersection of open sets,  $U \in \mathcal{T}_1$ . Hence,  $\mathcal{T}_1 \subseteq \mathcal{T}$ , a contradiction. Hence,  $\mathcal{T}$  is the unique coarsest topology of  $A$  under which each of the  $f_\alpha$  is open.  $\square$

(b) Let

$$\mathcal{S}_\beta = \{f_\beta^{-1}(U_\beta) \mid U_\beta \text{ is open in } X_\beta\},$$

and let  $\mathcal{S} = \cup \mathcal{S}_\beta$ . Show that  $\mathcal{S}$  is a subbasis for  $\mathcal{T}$ .

*Proof.* As we've constructed it,  $\mathcal{T}$  is the topology generated by the subbasis  $\mathcal{S}$ .  $\square$

(c) Show that a map  $g : Y \rightarrow A$  is continuous relative to  $\mathcal{T}$  if and only if each map  $f_\alpha \circ g$  is continuous.

*Proof.* Suppose, first of all, that  $g$  is continuous. Then, since each  $f_\alpha$  is continuous,  $f_\alpha \circ g$  is continuous, since the composition of continuous functions is continuous.

On the other hand, suppose that each  $f_\alpha \circ g$  is continuous. To show  $g$  is continuous, it suffices to show that, for each subbasis element  $\mathcal{S}_\beta$  of  $\mathcal{T}$ ,

$$g^{-1}(\mathcal{S}_\beta)$$

is open in  $Y$ . However, each  $\mathcal{S}_\beta = f_\beta^{-1}(U_\beta)$  for open  $U_\beta \subseteq X_\beta$ . Furthermore, since each  $f_\alpha \circ g$  is continuous,

$$g^{-1}(\mathcal{S}_\beta) = g^{-1}(f_\beta^{-1}(U_\beta)) = (f_\beta \circ g)^{-1}(U_\beta)$$

is open in  $Y$ . Thus, we can conclude that  $g$  is continuous.  $\square$

(d) Let  $f : A \rightarrow \prod X_\alpha$  be defined by the equation

$$f(a) = (f_\alpha(a))_{\alpha \in J};$$

let  $Z$  denote the subspace  $f(A)$  of the product space  $\prod X_\alpha$ . Show that the image under  $f$  of each element of  $\mathcal{T}$  is an open set of  $Z$ .

1

Let  $X$  be a topological space.  $X \subseteq X$  is said to be *dense* in  $X$  if  $\overline{S} = X$ .

(a) Let  $X$  be an infinite set with the finite complement topology. Show that any infinite subset  $S$  of  $X$  is dense in  $X$ .

*Proof.* Since  $X$  is an infinite set, by definition we know that the closed subsets of  $X$  are the finite subsets of  $X$  and  $X$  itself. Specifically, the only infinite closed subset of  $X$  is  $X$  itself. Let  $S$  be an infinite subset of  $X$ . Then

$$S \subseteq \overline{S}.$$

Hence, the closed set  $\overline{S}$  must be infinite; the only infinite closed subset is  $X$ , so  $\overline{S} = X$ . Hence,  $S$  is dense in  $X$ .  $\square$

(b) Now suppose  $f, g : X \rightarrow Y$  are continuous functions between two topological spaces, for which there is a dense subset  $S \subseteq X$  such that  $f(x) = g(x)$  for any  $x \in S$ . Show that if  $Y$  is Hausdorff, then  $f(x) = g(x)$  for all  $x \in X$ .

*Proof.* Suppose not. Let  $x \in X \setminus S$  such that  $f(x) \neq g(x)$ . Since  $Y$  is Hausdorff, this means that there exist disjoint open neighborhoods  $U$  and  $V$  of  $f(x)$  and  $g(x)$ , respectively. Hence, since  $f$  and  $g$  are continuous, there exist open neighborhoods  $U_x$  and  $V_x$  of  $x$  such that  $U_x \subseteq f^{-1}(U)$  and  $V_x \subseteq g^{-1}(V)$ . Clearly,

$$f(U_x) \cap g(V_x) = \emptyset$$

so  $f(a) \neq g(a)$  for  $a \in U_x \cap V_x$ . However,  $x \in U_x \cap V_x$ , so

$$U_x \cap V_x \neq \emptyset.$$

Since  $S$  is dense in  $X$ ,  $x$  is a limit point of  $S$ , so

$$S \cap (U_x \cap V_x) \neq \emptyset.$$

Let  $s \in S \cap (U_x \cap V_x)$ . Then  $f(s) = g(s)$ , a contradiction. From this contradiction, we can conclude that, in fact,  $f(x) = g(x)$  for all  $x \in X$ .  $\square$

2

Let  $A$  and  $B$  be homeomorphic subsets of  $\mathbb{R}^n$ . If  $A$  is closed in  $\mathbb{R}^n$  does it follow that  $B$  is also closed in  $\mathbb{R}^n$ ? For each dimension in which this is true please provide a proof and in each dimension for which it is false please provide a counterexample.

No, this does not hold true for any dimension. In general, let  $A = \mathbb{R}^n$  and

$$B = (-\pi/2, \pi/2) \times \mathbb{R}^{n-1}.$$

Clearly,  $A$  is closed in  $\mathbb{R}^n$  and  $B$  is open in  $\mathbb{R}^n$ . However, the map

$$f : A \rightarrow B$$

defined by

$$f(x) = f(x_1, \dots, x_n) = (\tan^{-1} x_1, x_2, \dots, x_n)$$

is a homeomorphism from  $A$  to  $B$ . To see that  $f$  is continuous, we need only note that its coordinate functions (namely  $\tan^{-1}$  and the identity map) are continuous. Furthermore,  $\tan$  is continuous on  $(-\pi/2, \pi/2)$ , so

$$f^{-1}(x) = f^{-1}(x_1, \dots, x_n) = (\tan x_1, x_2, \dots, x_n)$$

is continuous, since its coordinate functions are.

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