TOPOLOGY HW 5

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28.2

Show that $[0,1]$ is not limit point compact as a subspace of $\mathbb{R}_\ell$.

Proof. Define the infinite subset $A = \{1 - 1/k | k \in \mathbb{N}\} \subset [0,1]$. We want to demonstrate that A has no limit points in $\mathbb{R}_\ell$. First, we show that no element of A is a limit point of A. To see this, let $n \in \mathbb{N}$. Then $1 - 1/n \in A$.

$$1 - 1/n \in \left[1 - \frac{1}{n}, 1 - \frac{1}{n+1}\right),$$

which is an open set in $[0,1]$ as a subspace of $\mathbb{R}_\ell$. Since $\left[1 - \frac{1}{n}, 1 - \frac{1}{n+1}\right)$ intersects A only at $1 - \frac{1}{n}$, we see that $1 - 1/n$ is not a limit point of A.

Now, let $x \in (0,1), x \notin A$. Then $x \in \left[1 - \frac{1}{j}, 1 - \frac{1}{j+1}\right) = U_x$ for some $j \in \mathbb{N}$. Since U_x intersects A only at $1 - 1/j$, we see that the open set $\left[x, 1 - \frac{1}{j+1}\right) \ni x$ does not intersect A. Hence, x is not a limit point of A.

We only need to show, then, that 1 is not a limit point of A. But this is clear, as $[1,2)$ is open in $\mathbb{R}_\ell$ and $[1,2) \cap [0,1] = \{1\}$. Therefore, $\{1\}$ is a neighborhood of 1 which certainly does not intersect A, so 1 is not a limit point of A.

Since A is an infinite subset of $[0,1]$ with no limit points, we see that $[0,1]$ is not limit point compact as a subspace of $\mathbb{R}_\ell$. □

28.3

Let X be limit point compact.

(a) If $f : X \to Y$ is continuous, does it follow that $f(X)$ is limit point compact?

Answer: Yes. Let A be an infinite subset of $f(X)$. Then $A = \{f(x) | x \in B\}$ where $B \subseteq X$ is infinite. Since X is limit point compact, B has a limit point b. Let V_b be a neighborhood of $f(b)$. Then, since f is continuous, there exists some neighborhood U_b of b such that $f(U_b) \subseteq V_b$. Since b is a limit point of B, there exists some $y \in B$ such that $y \neq b, y \in U_b$. Thus,

$$f(y) \in f(U_b) \subseteq V_b,$$

so, since $f(y) \in A$, V_b intersects A at some point other than $f(b)$. Since our choice of neighborhood for $f(b)$ was arbitrary, we conclude that every neighborhood of $f(b)$ intersects A somewhere other than $f(b)$, meaning $f(b)$

is a limit point of A. Since our choice of infinite subset A was arbitrary, we conclude that every infinite subset of $f(X)$ has a limit point, so $f(X)$ is limit point compact.

♣

(b) If A is a closed subset of X, does it follow that A is limit point compact?

Answer: Yes. Let $Y \subseteq A \subseteq X$ be infinite. Then, since X is limit point compact, Y has a limit point $x \in X$. That is to say that any open neighborhood of x intersects Y at some point $y \neq x$. Since $y \in Y \subseteq A$, x is a limit point of A. However, since A is closed in X, A contains all of its limit points, so $x \in A$. Hence, Y has a limit point in A. Since our choice of infinite set Y was arbitrary, we conclude that every infinite subset of A has a limit point in A, so A is limit point compact.

♣

(c) If X is a subspace of the Hausdorff space Z, does it follow that X is closed in Z?

Answer: No. By Theorem 17.11, $\overline{S}_\Omega$ is Hausdorff in the order topology and, as we saw in Example 2, S_Ω is limit point point compact in $\overline{S}_\Omega$. However, S_Ω is not closed in $\overline{S}_\Omega$ since it does not contain Ω, which is a limit point of S_Ω.

♣

29.1

Show that the rationals $\mathbb{Q}$ are not locally compact.

Proof. Suppose $\mathbb{Q}$ is locally compact. Then there exists a compact subspace of $\mathbb{Q}$ containing a basic neighborhood $(a, b) \cap \mathbb{Q}$ of 0. Let x be an irrational element of (a, b). Then there exists a cauchy sequence $\{x_j\}$ converging to x such that each $x_j \in \mathbb{Q} \cap (a, b)$. Then the only limit point of the sequence of x_j's is x. However, $x \notin \mathbb{Q}$, so $\{x_j\}$, when viewed as a set, has no limit points in $\mathbb{Q}$ and, therefore, none in C. Hence, C is not limit point compact. Since $\mathbb{Q}$ is hausdorff, this means C is not compact. From this contradiction, we conclude that $\mathbb{Q}$ is not, in fact, locally compact. □

29.3

Let X be a locally compact space. If $f : X \to Y$ is continuous, does it follow that $f(X)$ is locally compact? What if f is both continuous and open?

If f is both continuous and open, then $f(X)$ is locally compact. To see this, let $y \in f(X)$. Then $f^{-1}(y) = x \in X$. So, since X is locally compact, there exists a compact subspace C of X that contains a neighborhood U of x. Since f is continuous, $f(C)$ is compact and since f is open, $f(U)$ is open in Y. $y \in f(U) \subseteq f(C)$. So $f(X)$ is locally compact at y. Since our choice of y was arbitrary, we conclude that $f(X)$ is locally compact at each of its points.

1

Let S^n denote the n-sphere.

(a) Show that S^n is connected for each $n \in \mathbb{N}$.

Proof. Let $n \in \mathbb{N}$ and define $f : \mathbb{R}^{n+1} \to S^n$ by

$$f(x) = \frac{x}{||x||}.$$

As seen in Example 5, pg. 156, this map is continuous and surjective, meaning $f(\mathbb{R}^{n+1}) = S^n$. Now $\mathbb{R}^{n+1}$ is connected and the continuous image of a connected space is connected, so S^n is connected. Since our choice of n was arbitrary, so we see that, for all $n \geq 1$, S^n is connected. □

(b) Show that no proper subset of S^n can be homeomorphic to S^n.

Proof. Suppose $A \subsetneq S^n$ such that $f : S^n \to A$ is a homeomorphism. Then A is compact. Let $p \in S^n - A$. Then $S^n - \{p\}$ is homeomorphic to $\mathbb{R}^n$ (since S^n is the one-point compactification of $\mathbb{R}^n$), so we can view A as a subset of $\mathbb{R}^n$. Since A homeomorphic to S^n, $A - \{f(p)\}$ is homeomorphic to $S^n - \{p\}$, which is homeomorphic to $\mathbb{R}^n$. By Brouwer's Invariance of Domain Theorem, then, $A - \{f(p)\}$ is open in $\mathbb{R}^n$. However, since A is compact and contained in $\mathbb{R}^n$, we know that A is closed. Clearly, we cannot subtract a single point from a compact subspace of $\mathbb{R}^n$ and get an open set. So $A - \{f(p)\}$ is not open. From this contradiction, we conclude that S^n is not homeomorphic to any proper subset of itself. □

2

(a) Show that if G is a group the element $e \in G$ is unique. It is known as the *identity element.*

Proof. Suppose not. Then there exist two elements $e_1, e_2 \in G$ such that $e_1 \neq e_2$ and for all $x \in G$,

$$e_1 x = x e_1 = 1 = e_2 x = x e_2.$$

Specifically,

$$e_2 = e_1 e_2 = e_1,$$

contradicting the fact that $e_1 \neq e_2$. From this contradiction, we conclude that the identity element $e \in G$ is unique. □

(b) Show that for any $g \in G$ there is a unique element $x \in G$ such that $x \cdot g = g \cdot x = e$. This element is called the *inverse* of g and is usually denoted as g^{-1}.

Proof. Let $g \in G$. By definition of a group, we know there must exist at least one element $h \in G$ such that

$$hg = gh = e.$$

Suppose there exist two such elements, h_1 and h_2. Then

$$h_1 = h_1 e = h_1(gh_2) = (h_1 g)h_2 = eh_2 = h_2.$$

Hence, the element $x \in G$ such that $gx = xg = e$ is unique. □

(c) Let $(G_1, \cdot)$ and $(G_2, \cdot)$ be two groups. A *group homomorphism* $f : G_1 \to G_2$ is a function such that $f(x \cdot y) = f(x) \cdot f(y)$ for any $x, y \in G_1$. A homomorphism $f : G_1 \to G_2$ is said to be an *isomorphism* if f is bijective and $f^{-1} : G_2 \to G_1$ is a homomorphism. Show that if $f : G_1 \to G_2$ is a group homomorphism, then $f(x^{-1}) = f(x)^{-1}$ for any $x \in G_1$.

Proof. Let $f : G_1 \to G_2$ be a group homomorphism and let $x \in G_1$. First, we show that, if e_1 and e_2 are the identity elements in G_1 and G_2, respectively, then $f(e_1) = e_2$. To see this, let $x \in G_1$. Then,

$$f(x) = f(xe_1) = f(x)f(e_1)$$

by the definition of a group homomorphism, so, since the identity element is unique, $f(e_1) = e_2$. Now, let $x \in G$. Then

$$f(x)^{-1} = f(x)^{-1}e_2 = f(x)^{-1}f(e_1) = f(x)^{-1}f(xx^{-1}) = (f(x)^{-1}f(x))f(x^{-1} = e_2 f(x^{-1}) = f(x^{-1}).$$

Since our choice of $x \in G$ was arbitrary, we conclude that $f(x^{-1}) = f(x)^{-1}$ for all $x \in G_1$. □

(d) A *subgroup* of a group $(G, \cdot)$ is a subset $H \subset G$ such that $e \in H$, $x \cdot y \in H$ for any $x, y \in H$ and $x^{-1} \in H$ for any $x \in H$. Now let $f : G_1 \to G_2$ be a homomorphism. The *kernel* of f is the set $\mathrm{Ker}(f) = f^{-1}(e_2)$, where $e_2 \in G_2$ is the identity element. Show that $\mathrm{Ker}(f)$ is a subgroup of G_1.

Proof. As we saw in part (c) above, we know that $e_1 \in \mathrm{Ker}(f)$. Now, suppose $x, y \in \mathrm{Ker}(f)$. Then

$$f(xy) = f(x)f(y) = e_2 e_2 = e_2,$$

so $xy \in G_2$. Finally, let $x \in \mathrm{Ker}(f)$. Then, since G_1 is a group, there exists $x^{-1} \in G_1$. By part (c), we know that

$$f(x^{-1}) = f(x)^{-1} = e_2^{-1} = e_2$$

so $x^{-1} \in \mathrm{Ker}(f)$. Since all the conditions of a subgroup are fulfilled, we conclude that $\mathrm{Ker}(f)$ is indeed a subgroup of G_1. □

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