TOPOLOGY FINAL

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1. Hausdorff Spaces

Let X be a Hausdorff space. Show that for any finite collection of points $\{x_1, \ldots, x_n\}$ in X there are neighborhoods $U_1, U_2, \ldots U_n$ of $x_1, x_2, \ldots x_n$ respectively such that $U_i \cap U_j = \emptyset$ if $i \neq j$.

Proof. We use induction on n, the number of points in the collection. Suppose $n = 2$, that is we have a collection $\{x_1, x_2\} \subseteq X$. Then, by the very definition of a Hausdorff space, there exist neighborhoods U_1, U_2 of x_1, x_2 respectively such that $U_1 \cap U_2 = \emptyset$.

Now, suppose that for any collection of k points, there exist neighborhoods fulfilling the above condition. Consider the collection $\{x_1, \ldots, x_{k+1}\} \subseteq U$. Now, $\{x_1, \ldots, x_k\}$ is a colletion of k points, so, by our induction hypothesis, there exist neighborhoods $U_1, \ldots, U_k$ of $x_1, \ldots, x_k$ respectively, such that $U_i \cap U_j = \emptyset$ if $i \neq j$. Let $m \in \{1, \ldots, k\}$. Then, since X is Hausdorff, there exist neighborhoods V_m, W_m of x_m and x_{k+1}, respectively, such that $V_m \cap W_m = \emptyset$. Then define $U'_m = U_m \cap V_m$. Then U'_m is an open neighborhood of x_m and $U'_m \cap W_m = \emptyset$. Furthermore, $U'_i \cap U'_j = \emptyset$ if $i \neq j$. Finally, define

$$U'_{k+1} = \cap_{i=1}^{k} W_i.$$

Since it is the finite intersection of open sets, U'_{k+1} is an open neighborhood of x_{k+1}. As we've constructed it, the collection $U'_1, \ldots, U'_{k+1}$ of open neighborhoods of $x_1, \ldots, x_{k+1}$, respectively, has the property that $U'_i \cap U'_j = \emptyset$ if $i \neq j$.

Therefore, by induction, we can find appropriate neighborhoods having this property for any finite collection of points in X. □

2. Implicit Function Theorem and Topological Manifolds

Use the implicit function theorem to show that as a *subspace* of $\mathbb{R}^{n+1}$ an n-surface M is locally homeomorphic to an open set of $\mathbb{R}^n$. That is, for each $p \in M$ there exists a neighborhood $O \subseteq M$ of p, an open set $U \subseteq \mathbb{R}^n$ and a homeomorphism $\phi : U \to O$.

Proof. Let M be an n-surface. Then there exists an open subset $W \subseteq \mathbb{R}^{n+1}$ and a C^∞ map $f : W \to \mathbb{R}$ such that $M = f^{-1}(c)$ for some $c \in \mathbb{R}$. We consider W as a subset of $\mathbb{R}^n \times \mathbb{R}$ and let $p \in M$. Hence, we can think of

p as being equal to (a, b), where $a \in \mathbb{R}^n$, $b \in \mathbb{R}$. Now, consider the 1×1 matrix

$$T = \left(\frac{\partial f}{\partial x_{n+1}} \right).$$

Certainly $\det T = \frac{\partial f}{\partial x_{n+1}} \neq 0$ so, since $f(p) = c$, by the implicit function theorem there exists $U \subseteq \mathbb{R}^n$ an open neighborhood of a and $V \subseteq \mathbb{R}$ an open neighborhood of b and a C^1 function $g : U \to V$ such that $(U \times V) \cap M = \{(x, g(x)) : x \in U\}$. Define $O := (U \times V) \cap M$. O is certainly open in M, since U and V are open, and a neighborhood of p, since $p = (a, b)$.

Hence, it suffices to show that

$$O = \{(x, g(x)) : x \in U\}$$

is homeomorphic to U. If we define $h : O \to U$ by

$$h : (x, g(x)) \mapsto x$$

then we see that h is clearly bijective and continuous. Furthermore, $h^{-1} : U \to O$ is just given by $h^{-1}(x) = (x, g(x))$. Since each of its component functions are continuous, h^{-1} is continuous. Therefore, since h is a bijective continuous map with continuous inverse, h is a homeomorphism between O and U. Since our choice of p was arbitrary, we conclude that M is locally homeomorphic to an open set of $\mathbb{R}^n$. □

3. Compactness & Connectedness

Let X be a compact Hausdorff space. Let $\mathcal{A}$ be a collection of *non-empty* closed connected subsets of X that is simply ordered by *proper* inclusion. Then $Y = \cap_{A \in \mathcal{A}} A$ is non-empty and connected.

Proof. First, we show that Y is non-empty. Let $A_1, \ldots, A_n \subseteq \mathcal{A}$. Then we can reorder the i's so that

$$A_1 \subsetneq A_2 \subsetneq \ldots \subsetneq A_n.$$

Then $\cap_1^n A_i = A_1 \neq \emptyset$, so $\mathcal{A}$ has the finite intersection property. Hence, since X is compact, Y is non-empty by Theorem 26.9.

Now, we show that Y is connected. Let $U, V \subseteq X$ be non-empty open sets such that $U \cap V = \emptyset$. Then, for all $A \in \mathcal{A}$,

$$C_A := A - ((A \cap U) \cup (A \cap V)) \neq \emptyset$$

since A is connected. Note that, since $(A \cap U) \cup (A \cap V)$ is open in A, C_A is closed in A. Since A is compact, this means C_A is compact and, since X is Hausdorff, closed in X. Now, let $\mathcal{C} = \{C_A : A \in \mathcal{A}\}$. Let $C_1, \ldots, C_n \subseteq \mathcal{C}$. Then, since $\mathcal{A}$ is simply ordered by proper inclusion, we can reorder the C_i's so that

$$C_1 \subsetneq C_2 \subsetneq \ldots \subsetneq C_n.$$

Then $\cap_1^n C_i = C_1 \neq \emptyset$, so $\mathcal{C}$ has the finite intersection property. Again using Theorem 26.9, we see that, therefore,

$$\cap_{A\in\mathcal{A}} C_A \neq \emptyset.$$

Hence,

$$Y - ((Y \cap U) \cup (Y \cap V)) = \cap_{A\in\mathcal{A}} C_A \neq \emptyset.$$

Since our choice of U, V was arbitrary, we conclude that Y cannot be separated. In other words, Y is connected. □

4. Covering Spaces & π_1

(a) Consider the following subspaces of $\mathbb{R}^2$: $E = (\mathbb{Z} \times \mathbb{R}) \cup (\mathbb{R} \times \mathbb{Z})$ (the infinite grid) and $B = \{(x, y) \in \mathbb{R}^2 : ||(x, y) - (-1, 0)|| = 1 \text{ or } ||(x, y) - (1, 0)|| = 1\}$ (the figure eight). Describe how E is a covering space for the space B. Drawing a picture along with your explanation may help. Also give an explicit formula for your covering map.

Answer: The explicit formula for a covering map is the following:

$$f((x, y)) = \begin{cases} 1 - e^{2\pi i y} & \text{if } (x, y) \in \mathbb{Z} \times \mathbb{R} \\ -e^{2\pi i x} + 1 & \text{if } (x, y) \in \mathbb{R} \times \mathbb{Z} \end{cases}$$

As we see, all points contained in $\mathbb{Z}\times\mathbb{Z}$ are mapped to the origin in the figure eight. Intuitively, a horizontal path starting at a point $(x_0, y_0) \in \mathbb{Z}\times\mathbb{Z})$ and traversing to the right will be projected onto a path traversing the righthand circle in the figure eight counterclockwise starting at the origin. Similarly, a vertical path starting at an integer point traversed upwards will project onto a counter-clockwise path around the lefthand circle of the figure eight, again starting at the origin.

♣

(b) Let E be the subspace of $\mathbb{R}^2$ considered above. Prove that $\pi_1(E)$ is non-trivial; that is, show that E is not simply connected.

Proof. Suppose that E is simply connected. Let f be the loop in B based at the origin that traverses the righthand circle once counter-clockwise, and let g be the loop in B based at the origin that traverses the lefthand circle once counter-clockwise.

Let γ be the loop in E that begins at the origin and traverses the unit square once counter-clockwise. Then

$$p \circ \gamma = f * g * f^{-1} * g^{-1},$$

so γ is the unique lift of $f * g * f^{-1} * g^{-1}$ to a path based at the origin. Since we assumed that E is simply connected, there exists a homotopy $H : [0, 1] \times [0, 1] \to E$ such that

$$\begin{aligned} H(s, 0) &= \gamma \\ H(s, 1) &= 0 \\ H(0, t) &= H(1, t) = 0 \end{aligned}$$

Now, define $H' := p \circ H$. Then

$$\begin{aligned} H'(s,0) &= p \circ H(s,0) = p \circ \gamma = f * g * f^{-1} * g^{-1} \\ H'(s,1) &= p \circ H(s,1) = p(0) = 0 \\ H'(0,t) &= H'(1,t) = p(0) = 0. \end{aligned}$$

Hence, since function composition preserves continuity, H' is a homotopy between $f*g*f^{-1}*g^{-1}$ and the constant map based at the origin. However, we showed in class that $[f]$ and $[g]$ do not commute in $\pi_1(B)$. That is to say that $[f]*[g] \neq [g]*[f]$, or that there is no homotopy between the loop $f*g*f^{-1}*g^{-1}$ and the constant map based at the origin.

From this contradiction, then, we conclude that γ is not homotopic to a constant loop, meaning $\pi_1(E)$ is not trivial. Therefore, E is not simply connected. □

5. Topological Groups

(a) Let G be a topological group and $H \leq G$ a closed subgroup. Show that if H and G/H are connected then G is connected.

Proof. Suppose, for the sake of contradiction, that G is not connected. That is to say, there exist non-empty open $A, B \subset G$ such that $A \cup B = G$ and $A \cap B = \emptyset$. Since the canonical projection $\pi : G \to G/H$ is surjective,

$$\pi(A) \cup \pi(B) = G/H.$$

Furthermore, since π is a quotient map, $\pi(A)$ and $\pi(B)$ are open in G/H. Hence, since G/H is connected, it must be the case that

$$\pi(A) \cup \pi(B) \neq \emptyset.$$

Let $cH \in \pi(A) \cup \pi(B)$ (where cH denotes the coset of H containing c). Then there exist $a \in A$ and $b \in B$ such that

$$\pi(a) = cH = \pi(b).$$

In other words, $a, b \in cH$. Since $A \cap B = \emptyset$, $(cH \cap A)$ and $(cH \cap B)$ form a separation of cH.

However, the multiplication by c is a continuous map and cH is, therefore, the continuous image of H under this map. Since H is connected, this means that cH is connected. From this contradiction, then, we conclude that there exists no separation of G, meaning G is connected. □

(b) Show that $SO(n)$ is connected for each $n \in \mathbb{N}$.

Proof. We prove this by induction on n. Since $SO(1) = \{1\}$, it is certainly clear that $SO(1)$ is connected. Now, assume that for $k \in \mathbb{N}$, $SO(k)$ is connected. Now, we proved in problem B of homework 8 that $SO(k+1)/SO(k)$ is homeomorphic to S^k, which we know to be connected. Therefore, since $SO(k) \leq SO(k+1)$ is connected and $SO(k+1)/SO(k)$ is connected, we conclude, by the result proved in (a) above, that $SO(k+1)$ is connected.

Therefore, we conclude, by induction, that $SO(n)$ is connected for all $n \in \mathbb{N}$. ☐

(c) Conclude that $SO(n)$ is the connected component of $O(n) = \{A \in M_n(\mathbb{R}) : A^t A = I\}$ containing the identity element

Proof. Certainly $SO(n) \subset O(n)$ and $I \in SO(n)$, since $\det I = 1$. Now, let $C \subset O(n)$ such that $SO(n) \subset C$. Then there is an element $A \in C$ such that $\det A = -1$ (since all elements of $O(n)$ have determinant equal to ± 1). Hence, the image of C under the determinant map is $\{1, -1\}$. Since $\{1, -1\}$ is disconnected and the determinant map is continuous, this implies that C must be disconnected as well. Hence, we conclude that $SO(n)$ is the connected component of $O(n)$ containing the identity element. ☐

6. Covering Spaces, Group Actions & π_1

(a) Suppose that $p : E \to B$ is a covering space and $f : X \to B$ is a continuous map where X is connected and B is Hausdorff. Show that if g_1 and g_2 are two lifts of f such that there exists an $x_0 \in X$ such that $g_1(x_0) = g_2(x_0)$, then $g_1(x) = g_2(x)$ for all $x \in X$.

Proof. Let g_1 and g_2 be lifts of f such that there exists $x_0 \in X$ with the property that

$$g_1(x_0) = g_2(x_0).$$

Let $C = \{x \in X : g_1(x) = g_2(x)\}$. Let U be a neighborhood of $f(x_0)$ that is evenly covered. Let V be the slice of $p^{-1}(U)$ that contains $g_1(x_0) = g_2(x_0)$. Then V is homeomorphic to U. Since g_1 and g_2 are continuous, $g_1^{-1}(V)$ and $g_2^{-1}(V)$ are open neighborhoods of x_0, so their intersection $g_1^{-1}(V) \cap g_2^{-1}(V)$ is an open neighborhood of x_0. Let $y_0 \in g_1^{-1}(V) \cap g_2^{-1}(V)$. Then $g_1(y_0) \in V$ and $g_2(y_0) \in V$. Furthermore, since g_1 and g_2 are lifts of f,

$$p(g_1(y_0)) = f(y_0) = p(g_2(y_0)).$$

Since p is a homeomorphism of V with U (specifically, it is injective), it must be the case that $g_1(y_0) = g_2(y_0)$. Hence, $y_0 \in C$. Since our choice of y_0 was arbitrary, we conclude that $g_1^{-1}(V) \cap g_2^{-1}(V) \subseteq C$. Therefore, C is open.

Now, let

$$D := X \backslash C = \{x \in X : g_1(x) \neq g_2(x)\}.$$

Let $x' \in D$ and let U' be an evenly covered neighborhood of $f(x')$. Let V_1 and V_2 be the slices of $p^{-1}(U')$ containing $g_1(x')$ and $g_2(x')$, respectively. Since $g_1(x') \neq g_2(x')$, we see that $V_1 \neq V_2$ and so $V_1 \cap V_2 = \emptyset$. Now, since g_1 and g_2 are continuous, $g_1^{-1}(V_1)$ and $g_2^{-1}(V_2)$ are open. Since both contain x', their intersection is non-empty, so we let $y' \in g_1^{-1}(V_1) \cap g_2^{-1}(V_2)$. Thus, $g_1(y') \in V_1$ and $g_2(y') \in V_2$. Since $V_1 \cap V_2 = \emptyset$,

$$g_1(y') \neq g_2(y').$$

Since our choice of y' was arbitrary, we conclude that

$$y' \in g_1^{-1}(V_1) \cap g_2^{-1}(V_2) \subseteq D.$$

Hence, since $y' \in g_1^{-1}(V_1) \cap g_2^{-1}(V_2)$ is open, we see that D is open.

Therefore, as $C = X \backslash D$, we see that C is both open and closed. Since X is connected and C is non-empty, this means that $C = X$. We conclude, then, that

$$g_1(x) = g_2(x)$$

for all $x \in X$. □

(b) Suppose G is a finite group which acts freely on a Hausdorff space X. Show that the action must be properly discontinuous.

Proof. Let $g \in G$ such that $g \neq e$ and let $x \in X$. Then, since G acts freely on X, $x \neq g(x)$. Hence, since X is Hausdorff, there exist open neighborhoods U and V of x and $g(x)$, respectively, such that $U \cap V = \emptyset$. Now, since the action of g is continuous, there exists an open neighborhood W of x such that

$$g(W) \subseteq V.$$

Since finite intersections of open sets are open, $U \cap W$ is an open neighborhood of x. Furthermore,

$$g(U \cap W) \subseteq g(W) \subseteq V,$$

so

$$(U \cap W) \cap g(U \cap W) = \emptyset.$$

Since our choices of g and x were arbitrary, we conclude that the action of G is properly discontinuous. □

(c) Suppose that E is a connected space and that G acts properly discontinuously on E. Let $\pi : E \to E/G$ be the quotient map. Show that $\pi : E \to E/G$ is a regular covering and that the group of deck transformations Δ is precisely G.

Proof. First, we show that the map induced by the action of an element $g \in G$ on X is a homeomorphism. If $x \in X$ and $y \in g^{-1}(x)$, then $g(y) = x$, so g's action is surjective. If $x, y \in X$ such that $g(x) = g(y)$, then certainly

$$x = e(x) = (g^{-1}g)(x) = g^{-1}(g(x)) = g^{-1}(g(y)) = (g^{-1}g)(y) = e(y) = y,$$

so g's action is injective. Furthermore, since the actions of both g and g^{-1} are just restrictions of the "action map" from $G \times X \to X$, both of their induced maps are continuous and are inverses to eachother. Hence, this map is a homeomorphism.

Now, let U be open in X. Then

$$\pi^{-1}(\pi(U)) = \bigcup_{g \in G} g(U).$$

Since each $g(U)$ is homeomorphic to U, this is a union of open sets, so $\pi^{-1}(\pi(U))$ is open. By definition of the quotient map, then, $\pi(U)$ is open. Hence, π is an open map.

Now, we show that $\pi : E \to G/E$ is a covering map. Note, first, that for any $g_0, g_1 \in G$, $g_0(U) \cap g_1(U) = \emptyset$ when $g_0 \neq g_1$, else $g_1^{-1}g_0(U)$ and U would not be disjoint. Let $x \in X$ and let U be a neighborhood of x such that $g_0(U)$ and $g_1(U)$ are disjoint whenever $g_0 \neq g_1$. Now, since π is open, $\pi(U)$ is open. Furthermore, as we have seen,

$$\pi^{-1}(\pi(U)) = \bigsqcup_{g \in G} g(U).$$

Let $g \in G$. Now, in order to show that $\pi(U)$ is openly covered by π, we must demonstrate that $g(U)$ and $\pi(U)$ are homeomorphic. Since we already know that π is open and continuous, we need only show that it provides a bijection. To that end, let $a, b \in U$ such that

$$\pi(g(a)) = \pi(g(b)).$$

In other words, there exists $g' \in G$ such that $g'(g(a)) = g'(g(b))$. Since we've already shown that the action of any element of G, including g', is injective, we conclude that $g(a) = g(b)$, so $\pi : g(U) \to \pi(U)$ is injective. Now, let $Gy \in \pi(U)$. Then $\pi(y) = Gy$, so π is surjective. Hence, since our choice of g was arbitrary, we conclude that, in fact, $g(U)$ is homeomorphic to $\pi(U)$ for all $g \in G$, meaning that $\pi(U)$ is openly covered by π. Since any element $Gx \in E/G$ is contained in such a neighborhood, we see that π covers E/G.

Now, to show that this covering is regular, we first show that G is actually equal to the group Δ of deck transformations of the covering. Let $g \in G$. Then

$$\pi(g(x)) = G(gx) = Gx = \pi(x),$$

so g is a deck transformation. Since our choice of g was arbitrary, we conclude that G is contained in the group of deck transformations.

On the other hand, let $D \in \Delta$. Then $\pi \circ D = \pi$, so, for any $x \in X$,

$$\pi(D(x)) = \pi(x) \Rightarrow \exists g \in G \text{ such that } D(x) = g(x).$$

Let $x_0 \in X$ and let $g_0 \in G$ such that $D(x_0) = g_0(x_0)$. Note that, if $f : E \to E$ is a lift of π, then $\pi \circ f = \pi$. In other words, the lifts of π are precisely the deck transformations of the covering. Since D and the map induced by the action of g_0 are both deck transformations, they are both lifts of π. We didn't use the hypothesis that B was Hausdorff in part (a) above, so we can apply that result here to conclude that, since $D(x_0) = g_0(x_0)$, D **is** the map induced by the action of g_0. In other words, every deck transformation is an element of G. Since we have demonstrated containment in both directions, we conclude that $G = \Delta$.

Now, to complete the argument, let $Gx \in E/G$ (where $x \in E$) and let $\pi^{-1}(Gx)$ be the fiber over Gx. Then

$$\pi^{-1}(Gx) = \{gx : g \in G\}$$

so we see that G acts transitively on the fiber. Since $G = \Delta$, we conclude that the group of deck transformations acts transitively on the fibers of the covering space, so $\pi : E \to E/G$ is, indeed, a regular covering. $\square$

(d) Let $S^{2n-1} = \{z = (z_1, \ldots, z_n) \in \mathbb{C}^n : ||z|| = 1\}$ $(n \geq 2)$ and fix a number $k \in \mathbb{N}$ and let $\epsilon = e^{\frac{i2\pi}{k}}$ be a k-th root of unity. The cyclic group of order k can then be expressed as $\mathbb{Z}_k = \{1, \epsilon, \ldots, \epsilon^{k-1}\}$. Now let $q_1, \ldots, q_n$ be integers which are relatively prime to k. We can then define an action of $\mathbb{Z}_k$ on S^{2n-1} via

$$\epsilon(z) = (\epsilon^{q_1} z_1, \ldots, \epsilon^{q_n} z_n).$$

The orbit space $S^{2n-1}/\mathbb{Z}_k$, denoted by $L(k; q_1, \ldots, q_n)$, is known as a **lens space**.

Based on the above, show that $\pi_1(L(k; q_1, \ldots, q_n)) \simeq \mathbb{Z}_k$ $(n \geq 2)$.

Proof. First, we show that $\mathbb{Z}_k$ acts freely on S^{2n-1}. Let $\epsilon^j \neq 1$ be an element of $\mathbb{Z}_k$ and let $z \in S^{2n-1}$. Suppose it were the case that $z = \epsilon^j(z)$. Then

$$z_1 = (\epsilon^j)^{q_1} z_1$$

meaning

$$1 = (\epsilon^j)^{q_1} = ((e^{\frac{i2\pi}{k}})^j)^{q_1} = e^{\frac{i2\pi(jq_1)}{k}}.$$

In turn, this implies that, for some $m \in \mathbb{N}$,

$$i2\pi m = \frac{i2\pi(jq_1)}{k} \Leftrightarrow mk = jq_1.$$

However, this is impossible, since q_1 and k are relatively prime. Thus, we see that $\mathbb{Z}_k$ acts freely on S^{2n-1}.

Since S^{2n-1} is Hausdorff and $\mathbb{Z}_k$ is finite, we conclude, based on part (b), that the action of $\mathbb{Z}_k$ on S^{2n-1} is properly discontinuous. This means, by part (c), that the group of deck transformations Δ is isomorphic to $\mathbb{Z}_k$. Hence, by the given theorem,

$$\mathbb{Z}_k \simeq \Delta \simeq (N_{\pi_1(B)}(p_*(\pi_1(E))))/p_*(\pi_1(E))$$

where $B = S^{2n-1}/\mathbb{Z}_k = L(k; q_1, \ldots, q_n)$ and $E = S^{2n-1}$. Now, as we've seen in class, $\pi_1(S^{2n-1})$ is trivial (since $n \geq 2$). Thus, its image in $\pi_1(B)$ is just the identity element of $\pi_1(B)$. Since the identity element of a group commutes with every element of the group, its normalizer is the entire group. Hence,

$$\mathbb{Z}_k \simeq (N_{\pi_1(B)}(p_*(\pi_1(E))))/p_*(\pi_1(E)) = \pi_1(B)/\{1\} = \pi_1(B).$$

Since $B = L(k; q_1, \ldots, q_n)$, we conclude that

$$\pi_1(L(k; q_1, \ldots, q_n)) \simeq \mathbb{Z}_k.$$

$\square$

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