GEOMETRY HW 10

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5.10.1

Introduce a metric on the projective plane P^2 so that the natural projection $\pi : S^2 \to P^2$ is a local isometry. What is the Gaussian curvature of such a metric?

Answer: First, we find parametrizations of the sphere S^2: we define $\phi_i : U \subset \mathbb{R}^2 \to S^2$ where $U = \{(u_1, u_2) \in \mathbb{R}^2 : u_1^2 + u_2^2 < 1\}$, where

$$\begin{aligned}
\phi_1(u_1,u_2) &= (u_1, u_2, \sqrt{1-(u_1^2+u_2^2)})\\
\phi_2(u_1,u_2) &= (u_1, u_2, -\sqrt{1-(u_1^2+u_2^2)})\\
\phi_3(u_1,u_2) &= (u_1, \sqrt{1-(u_1^2+u_2^2)}, u_2)\\
\phi_4(u_1,u_2) &= (u_1, -\sqrt{1-(u_1^2+u_2^2)}, u_2)\\
\phi_5(u_1,u_2) &= (\sqrt{1-(u_1^2+u_2^2)}, u_1, u_2)\\
\phi_6(u_1,u_2) &= (-\sqrt{1-(u_1^2+u_2^2)}, u_1, u_2).
\end{aligned}$$

Then, as we saw in Chapter 2, $\{\phi_i, U\}$ gives a parametrization of S^2. Furthermore, for all $i = 1, \ldots, 6$,

$$\phi_i(U) \cap A \cdot \phi_i(U) = \emptyset$$

where A is the antipodal map on the sphere. Therefore, for any $\pi \circ \phi_i$ is an injective map for each choice of i. Furthermore, if

$$\pi \circ \phi_i(U) \cap \pi \circ \phi_j(U) \neq \emptyset$$

then

$$(\pi \circ \phi_i)^{-1} \circ (\pi \circ \phi_j) = \phi_i^{-1} \circ \pi^{-1} \circ \pi \circ \phi_j = \phi_i^{-1} \circ \phi_j$$

and

$$(\pi \circ \phi_j)^{-1} \circ (\pi \circ \phi_i) = \phi_j^{-1} \circ \pi^{-1} \circ \pi \circ \phi_i = \phi_j^{-1} \circ \phi_i$$

each of which is differentiable, so we see that the family $\{\pi \circ \phi_i, U\}$ gives a parametrization of P^2, so P^2 is an abstract surface. In fact, it is clear that, since antipodal points on the sphere are identified in the projective plane, that ϕ_2, ϕ_4, ϕ_6 are redundant. Now, if $p \in S^2$, then let $\phi_i : U \to V \subset S^2$ be a coordinate chart containing p; then $\pi \circ \phi_i : U \to V' \subset P^2$ is a coordinate chart containing $\pi(p)$. Then

$$(\pi \circ \phi_i)^{-1} \circ \pi \circ \phi_i = \phi_i^{-1} \circ \pi^{-1} \circ \pi \circ \phi_i = \phi_i^{-1} \circ \phi_i = \mathrm{Id}$$

which is a differentiable map from U into itself, so we see that π is a differentiable map.

Now, let $p \in P^2$ and let $v_1, v_2 \in T_pP^2$. Then there exists $i = 1, 3, 5$ such that $\pi \circ \phi_i : U \to V \subset P^2$ is a coordinate chart containing p, where $p = \pi \circ \phi_i(u_0, v_0)$. Hence

$$\begin{aligned} v_1 &= a(\pi \circ \phi_i)_{u_1} + b(\pi \circ \phi_i)_{u_2} \\ v_2 &= \tilde{a}(\pi \circ \phi_i)_{u_1} + \tilde{b}(\pi \circ \phi_i)_{u_2}. \end{aligned}$$

Let $q = \phi_i(u_0, v_0)$ and let $w_1, w_2 \in T_qS^2$ such that

$$\begin{aligned} w_1 &= a(\phi_i)_{u_1} + b(\phi_i)_{u_2} \\ w_2 &= \tilde{a}(\phi_i)_{u_1} + \tilde{b}(\phi_i)_{u_2}. \end{aligned}$$

Then note that

$$\begin{aligned} d\pi_q w_1 = d\pi_q(a(\phi_i)_{u_1} + b(\phi_i)_{u_2}) = a d\pi_q(\phi_i)_{u_1} + b d\pi_q(\phi_i)_{u_2} = a(\pi \circ \phi_i)_{u_1} + b(\pi \circ \phi_i)_{u_2} = v_1 \\ d\pi_q w_2 = d\pi_q(\tilde{a}(\phi_i)_{u_1} + \tilde{b}(\phi_i)_{u_2}) = \tilde{a} d\pi_q(\phi_i)_{u_1} + \tilde{b} d\pi_q(\phi_i)_{u_2} = \tilde{a}(\pi \circ \phi_i)_{u_1} + \tilde{b}(\pi \circ \phi_i)_{u_2} = v_2. \end{aligned}$$

Now, define the metric $\langle , \rangle$ on P^2 by

$$\langle v_1, v_2 \rangle := \langle w_1, w_2 \rangle.$$

Now, let $q_0 \in S^2$ and let $z_1, z_2 \in T_{q_0}S^2$. Let $q_0 = \pi(p_0)$. Then

$$\langle d\pi_{q_0} z_1, d\pi_{q_0} z_2 \rangle = \begin{cases} \langle z_1, z_2 \rangle & q_0 \in \phi_1(U) \cup \phi_3(U) \cup \phi_5(U) \\ \langle dA_{q_0} z_1, dA_{q_0} z_2 \rangle & \text{otherwise.} \end{cases}$$

Since the antipodal map A is a an isometry, we see that

$$\langle d\pi_{q_0} z_1, d\pi_{q_0} z_2 \rangle = \langle z_1, z_2 \rangle$$

so π is a local isometry with respect to this metric.

Now we turn to the problem of determining the Gaussian curvature of the projective plane. Let $p_0 \in P^2$. Then there exists $i = 1, 3, 5$ such that $\pi \circ \phi_i : U \to V \subset P^2$ is a coordinate chart containing p_0, where $p_0 = \pi \circ \phi_i(u_0, v_0)$. Suppose, without loss of generality, that $i = 1$. Then, as we've designed the metric on P^2,

$$\begin{aligned} E = \langle (\pi \circ \phi_1)_{u_1}, (\pi \circ \phi_1)_{u_1} \rangle &= \langle (\phi_1)_{u_1}, (\phi_1)_{u_1} \rangle \\ &= \left\langle \left(1, 0, \tfrac{-u_1}{\sqrt{1-(u_1^2+u_2^2)}}\right), \left(1, 0, \tfrac{-u_1}{\sqrt{1-(u_1^2+u_2^2)}}\right) \right\rangle \\ &= 1 + \tfrac{u_1^2}{1-(u_1^2+u_2^2)} \\ &= \tfrac{1-u_2^2}{1-(u_1^2+u_2^2)}, \end{aligned}$$

$$\begin{aligned} F = \langle (\pi \circ \phi_1)_{u_1}, (\pi \circ \phi_1)_{u_2} \rangle &= \langle (\phi_1)_{u_1}, (\phi_1)_{u_2} \rangle \\ &= \left\langle \left(1, 0, \tfrac{-u_1}{\sqrt{1-(u_1^2+u_2^2)}}\right), \left(0, 1, \tfrac{-u_2}{\sqrt{1-(u_1^2+u_2^2)}}\right) \right\rangle \\ &= \tfrac{u_1 u_2}{1-(u_1^2+u_2^2)} \end{aligned}$$

and

$$\begin{aligned} G = \langle (\pi \circ \phi_1)_{u_2}, (\pi \circ \phi_1)_{u_2} \rangle \quad &= \langle (\phi_1)_{u_2}, (\phi_1)_{u_2} \rangle \\ &= \left\langle \left(0, 1, \tfrac{-u_2}{\sqrt{1-(u_1^2+u_2^2)}}\right), \left(0, 1, \tfrac{-u_2}{\sqrt{1-(u_1^2+u_2^2)}}\right) \right\rangle \\ &= 1 + \tfrac{u_2^2}{1-(u_1^2+u_2^2)} \\ &= \tfrac{1-u_1^2}{1-(u_1^2+u_2^2)}. \end{aligned}$$

We see that these are simply the coefficients of the first fundamental form on the given parametrization of the sphere. This implies that all of the Christoffel symbols of S^2 and P^2 will be identical, so we can see that the Gaussian curvature of P^2 at the point p_0 will be given by the Gaussian curvature of the sphere at $\phi_1(u_0, v_0) \in S^2$. The Gaussian curvature is given by the equation:

$$(\Gamma_{12}^1)_{u_1} - (\Gamma_{11}^1)_{u_2} + \Gamma_{12}^2 \Gamma_{12}^1 - \Gamma_{11}^2 \Gamma_{22}^1 = FK.$$

However, since the sphere lies in $\mathbb{R}^3$, we don't need to calculate its Gaussian curvature intrinsically; we are free to use the second fundamental form and we know the end result will agree with the intrinsic calculation. To that end, we first note that

$$\begin{aligned} (\phi_1)_{u_1u_1} &= \left(0, 0, \tfrac{-1}{\sqrt{1-(u_1^2+u_2^2)}} - \tfrac{u_1^2}{(1-(u_1^2+u_2^2))^{3/2}}\right), \\ (\phi_1)_{u_1u_2} &= \left(0, 0, \tfrac{-u_1u_2}{(1-(u_1^2+u_2^2))^{3/2}}\right), \\ (\phi_1)_{u_2u_2} &= \left(0, 0, \tfrac{-1}{\sqrt{1-(u_1^2+u_2^2)}} - \tfrac{u_2^2}{(1-(u_1^2+u_2^2))^{3/2}}\right). \end{aligned}$$

Now, the Gauss map N is simply given by $(u_1, u_2, \sqrt{1-(u_1^2+u_2^2)})$, so

$$\begin{aligned} e &= \langle N, (\phi_1)_{u_1u_1} \rangle = -1 - \tfrac{u_1^2}{1-(u_1^2+u_2^2)} \\ f &= \langle N, (\phi_1)_{u_1u_2} \rangle = \tfrac{-u_1u_2}{1-(u_1^2+u_2^2)} \\ g &= \langle N, (\phi_1)_{u_2u_2} \rangle = -1 - \tfrac{u_2^2}{1-(u_1^2+u_2^2)} \end{aligned} \quad .$$

Hence,

$$\begin{aligned} eg - f^2 &= \left(-1 - \tfrac{u_1^2}{1-(u_1^2+u_2^2)}\right)\left(-1 - \tfrac{u_2^2}{1-(u_1^2+u_2^2)}\right) - \tfrac{u_1^2u_2^2}{(1-(u_1^2+u_2^2))^2} \\ &= 1 + \tfrac{u_1^2+u_2^2}{1-(u_1^2+u_2^2)} + \tfrac{u_1^2u_2^2}{(1-(u_1^2+u_2^2))^2} - \tfrac{u_1^2u_2^2}{(1-(u_1^2+u_2^2))^2} \\ &= 1 + \tfrac{u_1^2+u_2^2}{1-(u_1^2+u_2^2)} \\ &= \tfrac{1}{1-(u_1^2+u_2^2)} \\ &= \tfrac{1-(u_1^2+u_2^2)}{(1-(u_1^2+u_2^2))^2}. \end{aligned}$$

On the other hand, given our calculations for E, F and G above,

$$\begin{aligned} EG - F^2 &= \left(\tfrac{1-u_2^2}{1-(u_1^2+u_2^2)}\right)\left(\tfrac{1-u_1^2}{1-(u_1^2+u_2^2)}\right) - \tfrac{u_1^2u_2^2}{(1-(u_1^2+u_2^2))^2} \\ &= \tfrac{(1-u_2^2)(1-u_1^2)}{(1-(u_1^2+u_2^2))^2} - \tfrac{u_1^2u_2^2}{(1-(u_1^2+u_2^2))^2} \\ &= \tfrac{1-(u_1^2+u_2^2)}{(1-(u_1^2+u_2^2))^2}. \end{aligned}$$

Hence,

$$K = \frac{eg - f^2}{EG - F^2} = 1,$$

as we would expect. Therefore, we see that the Gaussian curvature of P^2 at the point p_0 is 1.

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