GEOMETRY HW 3

CLAY SHONKWILER

2.4.13

A *critical point* of a differentiable functoin $f : S \to \mathbb{R}$ is defined on a regular suface S as a point $p \in S$ such that $df_p = 0$.

(a) Let $f : S \to \mathbb{R}$ be given by $f(p) = |p - p_0|, p \in S, p_0 \notin S$. Show that $p \in S$ is a critical point of f if and only if the line joining p to p_0 is normal to S at p.

Proof. Let $\alpha(t)$ be a curve on S with $\alpha(0) = p$, $\alpha'(0) = w$. Then p is a critical point if and only if

$$\begin{aligned} 0 = df_p(w) &= \frac{d}{dt}||p - p_0|||_{t=0} \\ &= \frac{d}{dt}(||\alpha(0) - p_0||)|_{t=0} \\ &= \frac{d}{dt}(\langle \alpha(0) - p_0, \alpha(0) - p_0\rangle^{1/2})|_{t=0} \\ &= \frac{-1}{2\langle \alpha(0) - p_0, \alpha(0) - p_0\rangle^{1/2}} \langle \alpha'(0), \alpha(0) - p_0\rangle \langle \alpha(0) - p_0, \alpha'(0)\rangle \\ &= \frac{-1}{2||p - p_0||} \langle w, p - p_0\rangle \langle p - p_0, w\rangle \\ &= \frac{-2\langle w, p - p_0\rangle}{2||p - p_0||} \\ &= \frac{-\langle w, p - p_0\rangle}{||p - p_0||} \end{aligned}$$

Hence, we see that it must be the case that $\langle w, p - p_0\rangle = 0$. Since this implies that the vector on the line joining p and p_0 is orthogonal to w. Since $w \in T_p(S)$ and our choice of α was arbitrary, we see that the line joining p and p_0 is normal to S at p if and only if p is a critical point of f. □

(b) Let $h : S \to \mathbb{R}$ be given by $h(p) = \langle p, v\rangle$, where $v \in \mathbb{R}^3$ is a unit vector. Show that $p \in S$ is a critical point of h if and only if v is a normal vector of S at p.

Proof. Let $\alpha(t)$ be a curve on S with $\alpha(0) = p$ and $\alpha'(0) = w$. Then p is a critical point of h if and only if

$$\begin{aligned} 0 = dh_p(w) &= \frac{d}{dt}(\langle \alpha(t), v\rangle)|_{t=0} \\ &= \langle \alpha'(0), v\rangle \\ &= \langle w, v\rangle \end{aligned}$$

Therefore, v and w are perpindicular. Since our choice of α was arbitrary and $w \in T_pS$, we see that v must be perpindicular to T_pS, meaning v is a normal vector of S at p if and only if p is a critical point of h. □

2.4.17

Two regular surfaces S_1 and S_2 intersect *transversally* if whenever $p \in S_1 \cap S_2$ then $T_p(S_1) \neq T_p(S_2)$. Prove that if S_1 intersects S_2 transversally, then $S_1 \cap S_2$ is a regular curve.

Proof. First, we need the following lemma:

Lemma 0.1. *Suppose $f : U \subseteq \mathbb{R}^3 \to \mathbb{R}^2$ is a differentiable function and $(a, b) \in f(U)$ is a regular value of f. Then $f^{-1}(a, b)$ is a regular curve in $\mathbb{R}^3$.*

Proof. Suppose f is as above, where

$$f(x, y, z) = (f_1(x, y, z), f_2(x, y, z))$$

and let $p = (x_0, y_0, z_0)$ be an element of $f^{-1}(a, b)$. Then, since (a, b) is a regular value, it is possible to assume, by renaming axes if necessary, that the minor determinant

$$\begin{vmatrix} \frac{\partial f_1}{\partial y}(p) & \frac{\partial f_1}{\partial z} \\ \frac{\partial f_2}{\partial y} & \frac{\partial f_2}{\partial z} \end{vmatrix} \neq 0.$$

Now, define $F : U \subset \mathbb{R}^3 \to \mathbb{R}^2$ such that

$$F(x, y, z) = (x, f_1(x, y, z), f_2(x, y, z)).$$

Let (u, v, t) indicate the coordinates of a point in the image of F. Then

$$dF_p = \begin{pmatrix} 1 & 0 & 0 \\ \frac{\partial f_1}{\partial x}(p) & \frac{\partial f_1}{\partial y}(p) & \frac{\partial f_1}{\partial z} \\ \frac{\partial f_2}{\partial x}(p) & \frac{\partial f_2}{\partial y} & \frac{\partial f_2}{\partial z} \end{pmatrix},$$

so

$$\det dF_p = \begin{vmatrix} \frac{\partial f_1}{\partial y}(p) & \frac{\partial f_1}{\partial z} \\ \frac{\partial f_2}{\partial y} & \frac{\partial f_2}{\partial z} \end{vmatrix} \neq 0.$$

Therefore, by the Inverse Function Theorem, there exist neighborhoods V and W of p and $F(p)$, respectively, such that $F : V \to W$ is a diffeomorphism. Hence, the coordinate functions of F^{-1}, namely

$$x = u, \quad y = g(u, v, t), \quad z = h(u, v, t),$$

are differentiable. In particular,

$$y = g(u, a, b) = \overline{g}(u) \text{ and } z = h(u, a, b) = \overline{h}(u)$$

are differentiable functions. Now, if we let $I \subset W \cap \{(u, v, t) : v = a, t = b\}$ be an interval such that $p \in F^{-1}(I)$, then $F^{-1}|_I$ is a diffeomorphism of I with a subset of $f^{-1}(a, b) \cap V$. Since our choice of p was arbitrary, we see that $f^{-1}(a, b)$ is, indeed, a regular curve. □

With this lemma in hand, we are now ready to attack the problem. Since S_1 and S_2 are regular surfaces, each is locally of the form $f^{-1}(a)$, where $f : U \subset \mathbb{R}^3 \to \mathbb{R}$ is a differentiable function and a is a regular value of f. Hence, suppose that S_1 is given by $f^{-1}(a)$ and S_2 by $g^{-1}(b)$ in a neighborhood of p. In this neighborhood, $S_1 \cap S_2$ is given by $F^{-1}(a,b)$, where

$$F : U \subset \mathbb{R}^3 \to \mathbb{R}^2$$

is defined by $F(q) := (f(q), g(q))$. Since S_1 and S_2 intersect transversally, the normal vectors (f_x, f_y, f_z) and (g_x, g_y, g_z) are not multiples of eachother, and so are certainly linearly independent. Hence,

$$dF_p = \begin{pmatrix} \frac{\partial f}{\partial x}(p) & \frac{\partial f}{\partial y}(p) & \frac{\partial f}{\partial z} \\ \frac{\partial g}{\partial x}(p) & \frac{\partial g}{\partial y} & \frac{\partial g}{\partial z} \end{pmatrix}$$

has a minor with non-zero determinant, so (a,b) is a regular value of F. Therefore, by our lemma, $S_1 \cap S_2$ is a regular curve. □

2.5.14

The *gradient* of a differentiable function $f : S \to \mathbb{R}$ is a differentiable map grad: $S \to \mathbb{R}^3$ which assigns to each point $p \in S$ a vector grad $f(p) \in T_p(S) \subset \mathbb{R}^3$ such that

$$\langle \text{grad} f(p), v\rangle = df_p(v) \quad \text{for all } v \in T_p(S).$$

Show that,

(a) If E, F, G are the coefficients of the first fundamental form in a paramerization $\phi : U \subset \mathbb{R}^2 \to S$, then grad f on $\phi(U)$ is given by

$$\text{grad} f = \frac{f_u G - f_v F}{EG - F^2}\Phi_1 + \frac{f_v E - f_u F}{EG - F^2}\Phi_v.$$

In particular, if $S = \mathbb{R}^2$ with coordinates x, y,

$$\text{grad} f = f_x e_1 + f_y e_2,$$

where $\{e_1, e_2\}$ is the canonical basis of $\mathbb{R}^2$.

Proof. Suppose $w \in T_pS$. Then $w = u_1\Phi_1 + u_2\Phi_2$ for some $u_1, u_2 \in \mathbb{R}$ and $df_p(w) = u_1 f_u + u_2 f_v$. Furthermore,

$$\begin{aligned} u_1 f_u + u_2 f_v &= df_p(w) \\ &= \langle \text{grad} f, w\rangle \\ &= \langle \alpha_1\Phi_1 + \alpha_2\Phi_2, u_1\Phi_1 + u_2\Phi_2\rangle \\ &= \alpha_1 u_1\langle\Phi_1,\Phi_1\rangle + \alpha_1 u_2\langle\Phi_1,\Phi_2\rangle + \alpha_2 u_1\langle\Phi_2,\Phi_1\rangle + \alpha_2 u_2\langle\Phi_2,\Phi_2\rangle \\ &= \alpha_1 u_1\langle\Phi_1,\Phi_1\rangle + (\alpha_1 u_2 + \alpha_2 u_1)\langle\Phi_1,\Phi_2\rangle + \alpha_2 u_2\langle\Phi_2,\Phi_2\rangle \end{aligned}$$

Substituting $E = \langle\Phi_1,\Phi_1\rangle$, $F = \langle\Phi_1,\Phi_2\rangle$ and $G = \langle\Phi_2,\Phi_2\rangle$, we see that

$$u_1 f_u + u_2 f_v = \alpha_1 u_1 E + (\alpha_1 u_2 + \alpha_2 u_1)F + \alpha_2 u_2 G = u_1(\alpha_1 E + \alpha_2 F) + u_2(\alpha_1 F + \alpha_2 G).$$

Now, we need to solve for α_1 and α_2.

$$u_1 f_u = u_1(\alpha_1 E \alpha_2 F)$$

so

$$\alpha_1 = \frac{f_u - \alpha_2 F}{E}.$$

Now,

$$u_2 f_v = u_2(\alpha_1 F + \alpha_2 G)$$

so, substituting for α_1, we see that

$$f_u F - \alpha_2 F^2 + \alpha_2 EG = f_v E.$$

Solving for α_2 and replacing that value into the expression for α_1, we see that

$$\text{grad} f = \alpha_1 \Phi_1 + \alpha_2 \Phi_2 = \frac{f_u G - f_v F}{EG - F^2}\Phi_1 + \frac{f_v E - f_u F}{EG - F^2}\Phi_2.$$

□

(b) If you let $p \in S$ be fixed and v vary in the unit circle $|v| = 1$ in T_pS, then $df_p(v)$ is maximum if and only if $v = \text{grad} f/|\text{grad} f|$.

Proof. By the definition of grad, we know that

$$df_p(v) = \langle \text{grad} f(p), v\rangle.$$

Applying the Schwarz inequality, we see that

$$df_p(v) = \langle \text{grad} f(p), v\rangle \leq ||\text{grad} f(p)||||v|| = ||\text{grad} f(p)||.$$

Now, if $v = \text{grad} f/|\text{grad} f|$, then

$$\begin{aligned} df_p(v) &= \langle \text{grad} f(p), \text{grad} f/||\text{grad} f||\rangle \\ &= \tfrac{1}{||\text{grad} f(p)||}\langle \text{grad} f(p), \text{grad} f(p)\rangle \\ &= \tfrac{||\text{grad} f(p)||^2}{||\text{grad} f(p)||} \\ &= ||\text{grad} f(p)||, \end{aligned}$$

the maximum value for $df_p(v)$. On the other hand, if $df_p(v)$ is maximized, then $df_p(v) = ||\text{grad} f(p)||$ (since we've just seen that this maximum is achieved), so, working backwords through the above chain of inequalities, we see that $v = \text{grad} f/||gradf||$. Therefore, we conclude that $df_p(v)$ is maximum if and only if $v = \text{grad} f/||\text{grad} f||$. □

(c) If grad $f \neq 0$ at all points of the *level curve* $C = \{q \in S; f(q) = \text{const}\}$, then C is a regular curve on S and grad f is normal to C at all points of C.

Proof. Based on the lemma we proved for problem 2.4.17 above, we see immediately that C is in fact a regular curve. Suppose it is given by a smooth function $\alpha : I \to \mathbb{R}^3$ such that $\alpha(0) = p$ and $\alpha'(0) = w$. Let $\beta = f \circ \alpha$. Then β is merely a constant map, so $\beta' \equiv 0$. Therefore

$$\langle \text{grad} f, w\rangle = df_p(w) = \beta'(0) = 0.$$

Since $w \in T_pS$, grad $f \neq 0$ at p and our choice of p was arbitrary, we see that, in fact, grad f is normal to C at all points of C. □

2.6.4

Prove that a regular, connected, orientable surface can have only two distinct orientations.

Proof. As we saw in class, any orientation determines a normal vector field N on the regular, connected, orientable surface S. Since $-N$ will also be a normal vector field on S, we see that there are at least two orientations for any such S.

To see that there are at most two, suppose there were three orientations for S, yielding three normal vector fields N_1, N_2, N_3. Now, define maps $\gamma_1, \gamma_2, \gamma_3 : S \to \{0, 2\}$ such that

$$\gamma_1(p) = ||N_1(p) - N_2(p)||, \gamma_2(p) = ||N_1(p) - N_3(p)||, \gamma_3(p) = ||N_2(p) - N_3(p)||,$$

where $N_i(p)$ is the element of N_i associated with p and $p \in S$. This is a well-defined functions, since each $N_i(p)$ is of length 1 for all $p \in S$. Now, since γ_i is merely the norm of a difference of continuous functions, γ_i is continuous for $i = 1, 2, 3$. Now, the continuous image of S is connected, since S is connected, so $Im(\gamma_i) = \{0\}$ or $Im(\gamma_i) = \{2\}$. Now, since N_1 and N_2 are associated with distinct orientations of S, it must be the case that $Im(\gamma_1) = \{2\}$, which implies $N_2(p) = -N_1(p)$ for all $p \in S$.

On the other hand, since N_1 and N_3 are associated with distinct orientations of S, it must be the case that $Im(\gamma_2) = \{2\}$ as well. This implies that $N_3(p) = -N_1(p)$ for all $p \in S$. Therefore, we see that

$$N_2(p) = -N_1(p) = N_3(p)$$

for all p, so N_2 and N_3 are the same normal vector field. However, this contradicts our assumption that N_1, N_2 and N_3 are associated with distinct orientations of S. From this contradiction, we conclude that there can be at most two orientations on any regular, connected, orientable surface. □

2.6.5

Let $\phi : S_1 \to S_2$ be a diffeomorphism.

(a) Show that S_1 is orientable if and only if S_2 is orientable.

Proof. Now, suppose $\{(U, \phi^{-1})\}$ gives a coherent atlas for S_1 that induces an orientation. Now, $\{(U, \phi^{-1} \circ f^{-1})\}$ gives an atlas on S_2, since f is a diffeomorphism.

Suppose $\psi := \phi \circ f : U \to V \subseteq S_2$ and $\psi' := f \circ \phi' : U' \to V' \subseteq S_2$ are two overlapping coordinate charts in this atlas. Then ψ induces an orientation on T_pS given by $[\Psi_1, \Psi_2]$ and ψ' induces an orientation given by $[\Psi_1', \Psi_2']$. Furthermore,

$$\Psi_1' = \Psi_1 \frac{\partial u}{\partial u'} + \Psi_2 \frac{\partial v}{\partial u'}$$

and

$$\Psi_2' = \Psi_1 \frac{\partial u}{\partial v'} + \Psi_2 \frac{\partial v}{\partial v'}.$$

Hence, the change of basis matrix is given by

$$\begin{pmatrix} \frac{\partial u}{\partial u'} & \frac{\partial v}{\partial u'} \\ \frac{\partial u}{\partial v'} & \frac{\partial v}{\partial v'} \end{pmatrix}.$$

However, this is precisely the change of basis between $[\Phi_1, \Phi_2]$ and $[\Phi_1', \Phi_2']$ in S_1. Since $\{(U, \phi^{-1})\}$ gives a coherent atlas on S_1, we see that

$$\frac{\partial(u,v)}{\partial(u',v')} > 0$$

so the orientations induced by $\psi : U \to V \subseteq S_2$ and $\psi' : U' \to V' \subseteq S_2$ are the same. Since our choice of ovelapping coordinate charts in the atlas $\{(U, \phi^{-1} \circ f^{-1})\}$ was arbitrary, we see that this atlas is coherent, so S_2 is orientable. □

(b) Let S_1 and S_2 be orientable and oriented. Prove that the diffeomorphism ϕ induces an orientation in S_2. Use the antipodal map of the sphere to show that this orientation may be distinct from the initial one.

Proof. In the proof of part (a) above, we demonstrated that an orientation of S_1 induces an orientation on S_2. □

1

A *skew-symmetric* bilinear form on a real vector space V is a bilinear map $\Omega : V \times V \to \mathbb{R}$ such that $\Omega(, v) = -\Omega(v, u)$ for any $u, v \in V$. Our goal is to prove the following theorem.

Theorem 1.1. *Let $\Omega : V \times V \to \mathbb{R}$ be a skew symmetric bilinear form. Then there exists a basis $B = \{u_1, \ldots, u_k, e_1, \ldots, e_n, f_1, \ldots, f_n\}$ for V such that*

(i) $\Omega(u_i, v) = 0$ for all $v \in V$.
(ii) $\Omega(e_i, e_j) = 0 = \Omega(f_i, f_j)$ for all $i, j = 1, \ldots, n$.
(iii) $\Omega(e_i, f_j) = \delta_{ij}$ for all $i, j = 1, \ldots, n$.

(a) Prove the above theorem.

Proof. Let $U = \{u \in V : \Omega(u, v) = 0 \forall v \in V\}$. Then, if $u, u' \in U$ and $a, b \in \mathbb{R}$, then

$$\Omega(au + bu', v) = a\Omega(u, v) + b\Omega(u', v) = 0 + 0 = 0$$

so $au + bu' \in U$. Hence, U is a subspace of V, so there exists a basis $\{u_1, \ldots, u_k\}$ for U. Now, define W to be a complement of U in V, meaning that $V = U \oplus W$.

Now, let $e_1 \in W$. Then there must exist $f_1' \in V$ such that $\Omega(e_1, f_1') \neq 0$, since, if there were no such f_1', then it would be the case that $\Omega(e_1, v) = 0$ for all $v \in V$, meaning that $e_1 \in U$ and, hence, $e_1 \notin W$. Now, since $V = U \oplus W$, $f_1' = u' + f_1$, where $u' \in U$, $f_1 \in W$. Hence

$$0 \neq \Omega(e_1, f_1') = \Omega(e_1, u' + f_1) = \Omega(e_1, u') + \Omega(e_1, f_1) = \Omega(e_1, f_1).$$

Suppose, $f_1 \in \text{Span}(e_1)$. Then $f_1 = ae_1$ for $a \in \mathbb{R}$, meaning

$$\Omega(e_1, f_1) = \Omega(e_1, ae_1) = a\Omega(e_1, e_1).$$

However, since Ω is skew symmetric,

$$\Omega(e_1, e_1) = -\Omega(e_1, e_1), \quad \text{so } \Omega(e_1, e_1) = 0.$$

Since $\Omega(e_1, f_1) \neq 0$, we see that, in fact, $f_1 \notin \text{Span}(e_1)$. If $\Omega(e_1, f_1) = \alpha$, then $\Omega(e_1, f_1/\alpha) = 1$, so we may assume, without loss of generality, that $\Omega(e_1, f_1) = 1$.

Now, define $W_1 := \text{Span}(e_1, f_1)$ and $W_1^\Omega := \{w \in W : \Omega(w, v) = 0 \forall v \in W_1\}$. Suppose $w \in W_1 \cap W_1^\Omega$. Then, since $w \in W_1$, $w = ae_1 + bf_1$ for some $a, b \in \mathbb{R}$. Furthermore, since $w \in W_1^\Omega$, for all $v \in W_1$, $v = ce_1 + df_1$ for some $c, d \in \mathbb{R}$ and

$$0 = \Omega(w, v) = \Omega(ae_1 + bf_1, ce_1 + df_1) = ac\Omega(e_1, e_1) + (ad - bc)\Omega(e_1, f_1) + bd\Omega(f_1, f_1).$$

Since this is true for all $v \in W_1$, we may as well let $c = a$ and $d = b$, which, if the above equality is to hold, implies that $a = b = 0$, meaning $w = 0$. Since our choice of w was arbitrary, we see that $W_1 \cap W_1^\Omega = 0$.

Now, let $f_v : W \to \mathbb{R}$ be given by

$$w \mapsto \Omega(w, v)$$

for $v \in W$. Since $f_{e_1}(f_1) = \Omega(e_1, f_1) = 1$ and $f_{f_1}(e_1) = \Omega(f_1, e_1) = -1$, we see that dim(Image($f_{e_1}$)) =dim(Image($f_{f_1}$)) = 1. Hence, by the rank theorem,

$$\dim(\text{Ker}(f_{e_1})) = \dim(\text{Ker}(f_{f_1})) = \dim W - 1.$$

Let us denote Ker f_v by S_v. If $w \in W_1^\Omega$, then

$$\Omega(w, e_1) = \Omega(w, f_1) = 0,$$

so $w \in S_{e_1} \cap S_{f_1}$, meaning $W_1^\Omega \subseteq S_{e_1} \cap S_{f_1}$. On the other hand, if $w' \in S_{e_1} \cap S_{f_1}$, then w' certainly cannot be a linear combination of e_1 and f_1, so $w' \in W_1^\Omega$. Therefore, $W_1^\Omega = S_{e_1} \cap S_{f_1}$. Now, we know that

$$\dim(S_{e_1} \cup S_{f_1}) = \dim(S_{e_1}) + \dim(S_{f_1}) - \dim(S_{e_1} \cap S_{f_1}).$$

Re-arranging to solve for $\dim(W_1^\Omega) = \dim(S_{e_1} \cap S_{f_1})$, we see that

$$\dim(W_1^\Omega) = \dim(S_{e_1}) + \dim(S_{f_1}) - \dim(S_{e_1} \cup S_{f_1}).$$

Now, $e_1 \in S_{e_1}$, but, since $\Omega(e_1, f_1) = 1$, $e_1 \notin S_{f_1}$. Since $\dim(S_{f_1} = \dim W - 1$, as we saw above, it must be true that

$$\dim(S_{e_1} \cup S_{f_1}) \geq \dim W.$$

Since $S_{e_1} \cup S_{f_1} \subseteq W$, we see that, in fact, $\dim(S_{e_1} \cup S_{f_1}) = \dim W$. Hence, pluggin into the equation for $\dim(W_1^\Omega)$ above, we see that

$$\begin{aligned} \dim(W_1^\Omega) &= \dim(S_{e_1}) + \dim(S_{f_1}) - \dim(S_{e_1} \cup S_{f_1}) \\ &= 2\dim W - 2 - \dim W \\ &= \dim W - 2. \end{aligned}$$

Since $\dim(W_1) + \dim(W_1^\Omega) = \dim W$ and $W_1 \cap W_1^\Omega = \{0\}$, we see that, in fact,

$$W = W_1 \oplus W_1^\Omega.$$

Now, repeat this same process on W_1^Ω, iterating until all elements of V have been exhausted. Each iteration yields a pair (e_i, f_i), meaning, at the end of this process, we will have a basis

$$\{u_1, \ldots, u_k, e_1, \ldots, e_n, f_1, \ldots, f_n\}$$

that fulfills the requirements of the theorem. □

(b) What is the matrix of Ω with respect to the basis B? That is, find a $(k+2n) \times (k+2n)$-matrix A such that with repsect to B we have $\Omega(u, v) = uAv^t$, where u, v are row vectors.

Answer: Let $A = (a_{ij})$. Then if we let

$$\{b_1, \ldots, b_{k+2n}\} = \{u_1, \ldots, u_k, e_1, \ldots, e_n, f_1, \ldots, f_n\}$$

as an ordered basis, then

$$a_{ij} = \Omega(b_i, b_j).$$

Hence, if $i \leq k$, then

$$a_{ij} = \Omega(b_i, b_j) = \Omega(u_i, b_j) = 0.$$

Similarly, if $j \leq k$, $a_{ij} = 0$. Now, if $k < i, j \leq k + n$, then

$$a_{ij} = \Omega(b_i, b_j) = \Omega(e_{i-k}, e_{j-k}) = 0 = \Omega(e_{j-k}, e_{i-k}) = \Omega(b_j, b_i) = a_{ji}.$$

Now, finally, if $k < i \leq k + n$ and $j > k + n$, then

$$a_{ij} = \Omega(b_i, b_j) = \Omega(e_{i-k}, f_{j-(k+n)}) = \left\{ \begin{array}{ll} 0 & i \neq j \\ 1 & i = j \end{array} \right.$$

Similarly, if $i > k + n$ and $k < j \leq k + n$, then

$$a_{ij} = \Omega(b_i, b_j) = \Omega(f_{i-(k+n)}, e_{j-k}) = \left\{ \begin{array}{ll} 0 & i \neq j \\ -1 & i = j \end{array} \right.$$

Hence, as a block matrix,

$$A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & A_1 \\ 0 & A_2 & 0 \end{pmatrix}$$

where A_1 and A_2 are $n \times n$ matrices such that

$$A_1 = \begin{pmatrix} 1 & 0 & \ldots & 0 \\ 0 & 1 & \ldots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & & & 1 \end{pmatrix}$$

and

$$A_2 = \begin{pmatrix} -1 & 0 & \dots & 0 \\ 0 & -1 & \dots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & & & -1 \end{pmatrix}.$$

♣

DRL 3E3A, University of Pennsylvania
E-mail address: shonkwil@math.upenn.edu