GEOMETRY HW 4

CLAY SHONKWILER

1.5.10

Consider the map

$$\alpha(t) = \begin{cases} (t, 0, e^{-1/t^2}), & t > 0 \\ (t, e^{-1/t^2}, 0), & t < 0 \\ (0, 0, 0), & t = 0 \end{cases}$$

(a) Prove that α is a differentiable curve.

Proof. If we denote $\alpha(t) = (x(t), y(t), z(t0)$, then it is clear that $x(t)$ is differentiable, with $x'(t) = 1$ for all t. Now, $y(t)$ and $z(t)$ are certainly differentiable for $t \neq 0$. To check that they are differentiable at zero as well, it suffices to show that

$$\lim_{t\to 0} \frac{d}{dt}(e^{-1/t^2}) = \lim_{t\to 0} \frac{2}{t^3} e^{-1/t^2} = 0.$$

To see this, we re-write and then apply L'Hopital's Rule twice:

$$\begin{aligned} \lim_{t\to 0} \tfrac{2}{t^3} e^{-1/t^2} &= \lim_{t\to 0} \frac{\frac{2}{t^3}}{e^{1/t^2}} \\ &= \lim_{t\to 0} \frac{\frac{-6}{t^4}}{\frac{-2}{t^3} e^{1/t^2}} \\ &= \lim_{t\to 0} \frac{\frac{3}{t}}{e^{1/t^2}} \\ &= \lim_{t\to 0} \frac{\frac{-3}{t^2}}{\frac{-2}{t^3} e^{1/t^2}} \\ &= \lim_{t\to 0} \frac{3t}{2e^{1/t^2}} \\ &= 0. \end{aligned}$$

Hence, we have agreement at $t = 0$, so α is differentiable and

$$\alpha'(t) = \begin{cases} (1, 0, \frac{2}{t^3} e^{-1/t^2}) & t \geq 0 \\ (1, \frac{2}{t^3} e^{-1/t^2}, 0) & t \leq 0 \end{cases}$$

□

(b) Prove that α is regular for all t and that the curvature $k(t) \neq 0$, for $t \neq 0$, $t \neq \pm\sqrt{2/3}$, and $k(0) = 0$.

Proof. As we can see from the explicit solution for $\alpha'(t)$ given above, $|\alpha'(t)| \geq 1$ for all t, so α is a regular curve. Furthermore, since

$$\frac{d}{dt}\left(\frac{2}{t^3}e^{-1/t^2}\right) = \left(\frac{4}{t^6} - \frac{6}{t^4}\right)e^{-1/t^2},$$

we see that

$$\alpha''(t) = \begin{cases} \left(0, 0, \left(\frac{4}{t^6} - \frac{6}{t^4}\right)e^{-1/t^2}\right) & t \geq 0 \\ \left(0, \left(\frac{4}{t^6} - \frac{6}{t^4}\right)e^{-1/t^2}, 0\right) & t \leq 0 \end{cases}$$

Hence, so long as $t \neq 0$ and

$$\frac{4}{t^6} - \frac{6}{t^4} \neq 0$$

(i.e. $t \neq \pm\sqrt{2/3}$), then we see that

$$k(t) = |\alpha''(t)| = \sqrt{\left(\left(\frac{4}{t^6} - \frac{6}{t^4}\right)e^{-1/t^2}\right)^2} = \left(\frac{4}{t^6} - \frac{6}{t^4}\right)e^{-1/t^2} \neq 0.$$

Furthermore, a L'Hopital argument similar to that given in (a) above demonstrates that $k(0) = 0$. □

(c) Show that the limit of the osculating planes as $t \to 0$, $t > 0$, is the plane $y = 0$ but that the limit of the osculating planes as $t \to 0$, $t < 0$, is the plane $z = 0$.

Proof. The osculating plane is the plane spanned by $\alpha'(t)$ and $\frac{\alpha''(t)}{k(t)}$, so it's normal vector is given by

$$\alpha'(t) \times \frac{\alpha''(t)}{k(t)}.$$

Note that, as $t \to 0$, $\alpha'(t) \to (1, 0, 0)$. When $t > 0$,

$$\frac{\alpha''(t)}{k(t)} = \frac{1}{k(t)}(0, 0, k(t)) = (0, 0, 1),$$

so

$$\alpha'(t) \times \frac{\alpha''(t)}{k(t)} \to (1, 0, 0) \times (0, 0, 1) = (0, -1, 0).$$

If (x, y, z) is a vector in the osculating plane, then

$$0 = \langle (x, y, z), (0, -1, 0) \rangle = -y,$$

so we see that the limit of the osculating planes is given by $y = 0$.

On the other hand, when $t < 0$,

$$\frac{\alpha''(t)}{k(t)} = \frac{1}{k(t)}(0, k(t), 0) = (0, 1, 0),$$

so

$$\alpha'(t) \times \frac{\alpha''(t)}{k(t)} \to (1, 0, 0) \times (0, 0, 1) = (0, 0, 1).$$

If (x, y, z) is a vector in this osculating plane, then

$$0 = \langle (x, y, z), (0, 0, 1) \rangle = z$$

so we see that the limit of the osculating planes is given by $z = 0$. □

(d) Show that τ can be defined so that $\tau \equiv 0$, even though α is not a plane curve.

Proof. Recall from the definition that

$$n(t) = \frac{\alpha''(t)}{|\alpha''(t)|}$$

and

$$b'(t) = \tau n.$$

Also, b is the normal vector to the osculating plane, so we see that, from part (c) above, for $0 < t < \sqrt{2/3}$ and $t > \sqrt{2/3}$,

$$b(t) = (0, -1, 0),$$

meaning that $b'(t) = (0, 0, 0)$ on this interval. On the other hand, again using our results from part (c), we know that, for $-\sqrt{2/3} < t < 0$ and $t < -\sqrt{2/3}$,

$$b(t) = (0, 0, 1),$$

meaning that $b'(t) = (0, 0, 0)$ on these intervals. Hence, we see that, anywhere $t \neq 0, t \neq \pm\sqrt{2/3}$,

$$(0, 0, 0) = b'(t) = \tau(0, -1, 0) = \tau(0, 0, 1),$$

which is to say that $\tau \equiv 0$ except possibly at these three points. However, since τ is continuous, we extend to these points by continuity to see that $\tau \equiv 0$ for all t. □

1.6.3

Show that the curvature $k(t) \neq 0$ of a regular parametrized curve $\alpha : I \to \mathbb{R}^3$ is the curvature at t of the plane curve $\pi \circ \alpha$, where π is the normal projection of α over the osculating plane at t.

Proof. We know that $k(s) = |\alpha''(s)|$, so it suffices to show that

$$k(s) = |\alpha''(s)| = |(\pi \circ \alpha)''(s)|$$

for all $t \in I$. Now, the osculating plane P_{s_0} at s_0 is simply the plane given by the span of $t(s_0) = \frac{\alpha'(s_0)}{|\alpha'(s_0)|}$ and $n(s_0) = \frac{\alpha''(s_0)}{|\alpha''(s_0)|}$. Since t and n are both unit vectors and are necessarily orthogonal, we know that the projection of α onto this plane is given by

$$\pi \circ \alpha(s) = \langle t(s_0), \alpha(s) \rangle t(s_0) + \langle n(s_0), \alpha(s) \rangle n(s_0).$$

Hence,

$$(\pi \circ \alpha)''(s) = \langle t, \alpha''(s) \rangle t + \langle n, \alpha''(s) \rangle n = \langle n, \alpha''(s) \rangle n = \frac{\langle \alpha''(s_0), \alpha''(s) \rangle}{|\alpha''(s_0)} n.$$

Hence, when $s = s_0$,

$$(\pi \circ \alpha)''(s_0) = \frac{\langle \alpha''(s_0), \alpha''(s) \rangle}{|\alpha''(s_0)} n = \frac{|\alpha''(s_0)|^2}{|\alpha''(s_0)|} n.$$

Therefore,

$$|(\pi \circ \alpha)''(s_0)| = ||\alpha''(s_0)|n| = |\alpha''(s_0)| = k(s_0).$$

Hence, we see that the curvature of α is the curvature of its projection onto the osculating plane. □

3.2.5

Show that the mean curvature H at $p \in S$ is given by

$$H = \frac{1}{\pi} \int_0^\pi k_n(\theta) d\theta,$$

where $k_n(\theta)$ is the normal curvature at p along a direction making an angle θ with a fixed direction.

Proof. By definitions, $H = \frac{k_1+k_2}{2}$, where k_1 and k_2 are the maximum and minimum normal curvatures, respectively. Also, $k_n(\theta) = k_1 \cos^2\theta + k_2 \sin^2\theta$. Therefore,

$$\begin{aligned}
\frac{1}{\pi}\int_0^\pi k_n(\theta)d\theta &= \frac{1}{\pi}\int_0^\pi k_1\cos^2\theta + k_2 \sin^w \theta d\theta \\
&= \frac{1}{\pi}\int_0^\pi k_1(1-\sin^2\theta) + k_2\sin^2\theta d\theta \\
&= \frac{1}{\pi}\int_0^\pi k_1 + \sin^2\theta(k_2 - k_1) d\theta \\
&= \frac{1}{\pi}\left[[k_1\theta]_0^\pi + (k_2-k_1)\int_0^\pi \sin^2\theta d\theta\right] \\
&= k_1 + \frac{k_2-k_1}{\pi}\int_0^\pi \frac{1-\cos 2\theta}{2} d\theta \\
&= k_1 + \frac{k_2-k_1}{\pi}\left[\theta/2 + \frac{\sin 2\theta}{4}\right]_0^\pi \\
&= k_1 + \frac{k_2-k_1}{2} \\
&= \frac{k_1+k_2}{2} \\
&= H.
\end{aligned}$$

□

3.2.8

Describe the region of the unit sphere covered by the image of the Gauss map of the following surfaces:

(a) Paraboloid of revolution $z = x^2 + y^2$.

(b) Hyperboloid of revolution $x^2 + y^2 - z^2 = 1$.

Answer: Consider the map $f : \mathbb{R}^3 \to \mathbb{R}$ given by $f(x, y, z) = x^2 + y^2 - z^2$. Then $df_p = (f_x, f_y, f_z) = (2x, 2y, -2z)$, which is nonzero so long as $(x, y, z) \neq 0$. Since $f(0) \neq 1$, we see that 1 is a regular value of f. Hence, we see that the hyperboloid S is a regular surface. Now, note that

$$||\nabla f|| = \sqrt{4x^2 + 4y^2 + 4z^2} = 2\sqrt{x^2 + y^2 + z^2} = 2\sqrt{(1 + z^2) + z^2} = 2\sqrt{(1 + 2z^2}.$$

Since the hyperboloid is a regular surface, we know that the normal vector field is given by

$$(x_0, y_0, z_0) = N = \frac{\nabla f(p)}{||\nabla f(p)||} = \frac{(2x, 2y, -2z)}{2\sqrt{1+2z^2}} = \left(\frac{x}{\sqrt{1+2z^2}}, \frac{y}{\sqrt{1+2z^2}}, \frac{z}{\sqrt{1+2z^2}}\right).$$

Hence, we see that when z is constant, the z-coordinate of N is also constant and the x- and y-coordinates of N are proportional to x and y, respectively. In other words, if a vector (x_0, y_0, z_0) is in the image of the Gauss map, then the entire latitudinal cirle given by

$$\{(x, y, z) \in S^2 : z = z_0\}$$

will also be in the image. Hence, to describe the image, we need only find the maximum and minimum possible values of z_0, which will, in turn, give the image of the Gauss map as a band around the equator of the sphere.

To do so, we can assume that $x = 0$. Then, since they are the coordinates of a point lying on the hyperbola, y and z satisfy the following relation:

$$y^2 - z^2 = 1.$$

Note that this implies that $y^2 \geq 1$. In other words, $z = \pm\sqrt{y^2-1}$, so we see that

$$z_0 = \frac{\mp\sqrt{y^2-1}}{\pm\sqrt{2y^2-1}}$$

As $y \to \infty$, we see that $z_0 \to \pm\sqrt{2}/2$ and that, furthermore, if we consider a single branch of the above expression, the possible values for z_0 start at zero and increase or decrease monotonically. Hence, $z_0 \in [0, \sqrt{2}/2)$. Based on our above argument that concluded that this solution holds for all x and y, we see that, in fact, the image of the hyperboloid under the Gauss map is given by

$$\{(x, y, z) \in S^2 : |z| < \sqrt{2}/2\}.$$

♣

(c) Catenoid $x^2 + y^2 = \cosh^2 z$.

Answer: Consider the function $f : \mathbb{R}^3 \to \mathbb{R}$ given by

$$f(x, y, z) = x^2 + y^2 - \cosh^2 z.$$

Then, recalling that $\cosh z = \frac{1}{2}(e^z + e^{-z})$, we see that

$$\nabla f = (f_x, f_y, f_z) = (2x, 2y, -\frac{1}{2}(e^2z - e^{-2z})).$$

Then we see that the only critical value of f occurs at the point $(0, 0, 0)$. Now, $f(0, 0, 0) = -1/2$, so 0 is a regular value of f. Hence, the catenoid S that we are considering is $f^{-1}(0)$, where 0 is a regular value. Therefore, S is a regular surface.

Thus, the normal vector field is given by

$$\begin{aligned}(x_0, y_0, z_0) = N_p &= \frac{\nabla f(p)}{||\nabla f(p)||} \\ &= \frac{(2x, 2y, -\frac{1}{2}(e^{2z} - e^{-2z}))}{\sqrt{4x^2+4y^2+\frac{1}{4}(e^{4z}+e^{-4z}-2)}} \\ &= \frac{(2x, 2y, -\frac{1}{2}(e^{2z} - e^{-2z}))}{\sqrt{4\cosh^2 z+\frac{1}{4}(e^{4z}+e^{-4z})-\frac{1}{2})}} \\ &= \frac{(2x, 2y, -\frac{1}{2}(e^{2z} - e^{-2z}))}{\sqrt{4\cosh^2 z+\frac{1}{2}\cosh 4z-\frac{1}{2}}}.\end{aligned}$$

Hence, as in part (b) above, when z is constant then z_0 is constant and x_0 and y_0 are proportional to x and y, respectively. In other words, as before, if (x_0, y_0, z_0) is in the image of the Gauss map, then the entire latitudinal circle given by

$$\{(x, y, z) \in S^2 : z = z_0\}$$

will also be in the image. Therefore, we need only determine the possible values of z_0 in order to completely describe the image of the Gauss map.

Now, let $(x, y, z) \in S$ and define $\alpha := \frac{1}{\sqrt{4\cosh^2 z+\frac{1}{2}\cosh 4z-\frac{1}{2}}}$. Then

$$(x_0, y_0, z_0) = \left(\frac{2x}{t}, \frac{2y}{z}, \frac{-\frac{1}{2}(e^{2z} - e^{-2z})}{t}\right).$$

Then

$$1 = x_0^2 + y_0^2 + z_0^2 = \frac{4x^2}{t^2} + \frac{4y^2}{t^2} + z_0^2 = 4\frac{\cosh^2 z}{t^2} + z_0^2.$$

In other words,

$$z_0 = \pm\left(1 - \frac{1}{t^2}\cosh^2 z\right)^{\frac{1}{2}}.$$

Now, we see that, it must be the case that

$$\begin{aligned}|z_0| &< \lim_{z\to\infty}\left(1 - \frac{1}{t^2}\cosh^2 z\right)^{\frac{1}{2}} \\ &= \lim_{z\to\infty}\left(1 - \frac{\cosh^2 z}{4\cosh^2 z+\frac{1}{2}\cosh 4z-\frac{1}{2}}\right)^{\frac{1}{2}} \\ &= \lim_{z\to\infty}\left(1 - \frac{\frac{1}{4}(e^{2z}+e^{-2z}+2)}{e^{2z}+e^{-2z}+2+\frac{1}{4}(e^{4z}-e^{-4z})-\frac{1}{2}}\right)^{\frac{1}{2}} \\ &= \lim_{z\to\infty}\left(1 - \frac{1}{4}\left(\frac{e^{2z}+e^{-2z}+2}{e^{2z}+e^{-2z}+\frac{3}{2}+\frac{1}{4}(e^{4z}-^{-4z})}\right)\right)^{\frac{1}{2}} \\ &= \lim_{z\to\infty}\left(1 - \frac{1}{4}\left(\frac{1+e^{-4z}+2e^{-2z}}{1+e^{-4z}+\frac{3}{2}e^{-2z}+\frac{1}{4}(e^{2z}-e^{-6z})}\right)\right)^{\frac{1}{2}} \\ &= \lim_{z\to\infty}\left(1 - \frac{1}{4}\frac{1}{1+\frac{1}{4}e^{2z}}\right)^{\frac{1}{2}} \\ &= (1-0)^{\frac{1}{2}} \\ &= 1,\end{aligned}$$

where we arrived at the fourth equality by multiplying numerator and denominator by e^{-2z}. Now, since z can be any element of $\mathbb{R}$, we see that z_0

achieves every value on the sphere except $z_0 = 1$ and $z_0 = -1$. Therefore, the image of the Gauss map is all of S^2 except the north and south poles.

♣

1

Suppose $F : \mathbb{R}^n \to \mathbb{R}^k$ is a C^∞-function and $r \in \mathbb{R}^k$ is a regular value of F; that is, for each $p \in F^{-1}(r)$ w have that $dF_p : T_p\mathbb{R}^n \to T_{F(p)}\mathbb{R}^k$ is onto. Show that for each $p \in S = F^{-1}(r)$ there exists $V \subseteq S$ open neighborhood of p in S and $\phi : U \subseteq \mathbb{R}^{n-k} \to V$ such that

(a) $\phi : U \to V \subset \mathbb{R}^n$ is differentiable.
(b) $\phi : U \to V$ is a homeomorphism.
(c) $d\phi_q : T_q\mathbb{R}^{n-k} \to T_{\phi(q)}\mathbb{R}^n$ is injective for any $q \in U$.

Proof. Denote $r = (r_1, \dots, r_k)$ and let $p \in F^{-1}(r)$. Since r is a regular value, the matrix

$$dF_p = \begin{pmatrix} \frac{\partial F_1}{\partial x_1} & \frac{\partial F_1}{\partial x_2} & \dots & \frac{\partial F_1}{\partial x_n} \\ \frac{\partial F_2}{\partial x_1} & \frac{\partial F_2}{\partial x_2} & \dots & \frac{\partial F_2}{\partial x_n} \\ \vdots & & \ddots & \vdots \\ \frac{\partial F_k}{\partial x_1} & \frac{\partial F_k}{\partial x_2} & \dots & \frac{\partial F_k}{\partial x_n} \end{pmatrix}$$

has a minor with nonzero determinant. Assume, without loss of generality, that the rightmost k columns are such a minor. Now, define $H : \mathbb{R}^n \to \mathbb{R}^n$ by

$$H(x_1, x_2, \dots, x_n) = (x_1, x_2, \dots, x_{n-k}, F_1(x_1, \dots, x_n), \dots, F_k(x_1, \dots, x_n)).$$

We will denote by $(u_1, \dots, u_n)$ the elements in $\mathbb{R}^3$ where H takes its values. Now,

$$dH_p = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 & \dots & 0 \\ \vdots & & \ddots & & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 & 0 & \dots & 0 \\ \frac{\partial F_1}{\partial p_1} & \frac{\partial F_1}{\partial p_2} & & \dots & & & \dots & \frac{\partial F_1}{\partial p_n} \\ \vdots & & & & & & & \vdots \\ \frac{\partial F_k}{\partial p_1} & \frac{\partial F_k}{\partial p_2} & & \dots & & & \dots & \frac{\partial F_k}{\partial p_n} \end{pmatrix}.$$

Then

$$\det dH_p = \begin{vmatrix} \frac{\partial F_1}{\partial p_{n-k+1}} & \dots & \frac{\partial F_1}{\partial p_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial F_k}{\partial p_{n-k+1}} & \dots & \frac{\partial F_k}{\partial p_n} \end{vmatrix} \neq 0.$$

By the Inverse Function Theorem, then, there exists a neighborhood V' of p and W of $F(p)$ such that $H : V' \to W$ is a diffeomorphism. Hence, the coordinate functions

$$x_1 = u_1, \dots, x_{n-k} = u_{n-k}, x_{n-k+1} = g_1(u_1, \dots, u_n), \dots, x_n = g_k(u_1, \dots, u_n)$$

of H^{-1} are differentiable. In particular, for $i = 1, \dots, k$,

$$x_{n-k+i} = g_i(u_1, \dots, u_{n-k}, r_1, \dots, r_k) = h_i(u_1, \dots, u_{n-k})$$

are differentiable functions defined on $\pi'(V')$, where π' is the projection of $F^{-1}(r)$ onto the set

$$U' = W \cap \{(u_1, \dots, u_{n-k}, r_1, \dots, r_k)\}.$$

Note that

$$H(F^{-1}(r) \cap V) = U',$$

so, in fact,

$$H|_{U'} : U' \to F^{-1}(r) \cap V' = V$$

is a diffeomorphism. Now, if we let $\pi : \mathbb{R}^n \to \mathbb{R}^n$ be the projection onto the set

$$A = \{(x_1, \dots, x_{n-k}, 0, 0, \dots, 0\},$$

then it is clear that $\pi|_{U'}$ is a diffeomorphism of U' with $\pi|_{U'}(U')$. Define $U := \pi|_{U'}(U')$. Then $\phi : U \to V$ given by

$$\phi(u_1, \dots, u_{n-k}) = (u_1, \dots, u_{n-k}, h_1(u_1, \dots, u_{n-k}), \dots, h_k(u_1, \dots, u_{n-k}))$$

is simply the composition $\phi = H|_{U'} \circ \pi_{U'}^{-1}$. Since both of the functions in the composition are diffeomorphisms, so is ϕ. Hence, $\phi : U \to V$ precisely fulfills the desired requirements. □

DRL 3E3A, University of Pennsylvania
E-mail address: shonkwil@math.upenn.edu