

## GEOMETRY HW 7

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### 4.5.1

Let  $S \subset \mathbb{R}^3$  be a regular, compact, orientable surface which is not homeomorphic to a sphere. Prove that there are points on  $S$  where the Gaussian curvature is positive, negative, and zero.

*Proof.* By Corollary 2 of the Gauss-Bonnet,

$$\int \int_S K d\sigma = 2\pi\chi(S).$$

Now, by Proposition 4, since  $S$  is not homeomorphic to the sphere, it cannot have Euler-Poincare characteristic 2. Therefore,

$$\chi(S) = 0, -2, \dots, -2n, \dots$$

Hence,

$$\int \int_S K d\sigma \leq 0,$$

so if we can demonstrate that  $K$  must attain a positive value on at least one point in  $S$ , then we can see immediately that  $K$  must attain a negative value at some point and therefore, since  $K$  is continuous, we can use the intermediate value theorem to conclude that  $K$  is also zero at some point. In other words, proving the following lemma suffices to prove the desired result:

**Lemma 0.1.** *If  $S \subset \mathbb{R}^3$  is a regular, compact, orientable surface, then  $S$  has an elliptic point.*

*Proof.* We know that taking the norm of any point  $p \in S$  produces a continuous function from  $S$  to  $\mathbb{R}$ . Since  $S$  is compact, the image of the norm is compact, and so has a maximum. Let  $p_0 \in S$  be a point at which the norm achieves a maximum. Then  $p_0$  lies in both  $S$  and the 2-sphere of radius  $|p|$ , denoted  $S^2_{|p_0|}$ . If  $\phi : U \rightarrow V \subseteq S$  is a coordinate chart of  $S$  containing  $p$  such that  $\phi(0, 0) = p_0$ , then certainly  $|q| \leq |p_0|$ , so we see that, for  $q \in V$ , either  $q \in S^2_{|p_0|}$  or  $q$  is in the interior of  $S^2_{|p_0|}$ . In either case, this implies that the intersection of  $S$  with  $S^2_{|p_0|}$  is tangent to  $S^2_{|p_0|}$ , so

$$T_{p_0}S = T_{p_0}S^2_{|p_0|}.$$

Since every point on the sphere is an elliptic point, we know, by proposition 1 of Section 3.2 that every point on the sphere in a neighborhood of  $p_0$  lies

on the same side of this tangent plane, and thus we can conclude that every point in  $V$  lies on the same side of  $T_{p_0}S$ . Now, if  $N(p_0)$  denotes the normal to  $p_0$ , then we can define the distance from a point  $q = \phi(u_1, u_2)$  to  $T_{p_0}S$  by

$$d(q) = d(\phi(u_1, u_2)) = \langle \phi(u_1, u_2) - \phi(0, 0), N(p) \rangle$$

Since each  $q$  lies on the same side of the tangent plane,  $d$  has a fixed sign. By the proof of Proposition 1 of Section 3.2, this implies that  $II_{p_0}$  has the same fixed sign, which in turn means that the Gaussian curvature of  $p_0$  on  $S$  is positive. Hence,  $p_0$  is an elliptic point of  $S$ .  $\square$

$\square$

#### 4.5.2

Let  $T$  be a torus of revolution. Describe the image of the Gauss map of  $T$  and show, without using the Gauss-Bonnet theorem, that

$$\int \int_T K d\sigma = 0.$$

Compute the Euler-Poincare characteristic of  $T$  and check the above result with the Gauss-Bonnet theorem.

**Answer:** The image of the Gauss map of  $T$  is all of  $S^2$ , the 2-sphere, since the image of each meridian of the torus under the Gauss map is a meridian of the sphere; ranging over all meridians of the torus covers all meridians of the sphere.

Now, in order to compute  $\int \int_T K d\sigma$ , we can make use of the equality:

$$\int \int_T Dd\sigma = \int \int_{\phi^{-1}(S)} K(u_1, u_2) \sqrt{EG - F^2} du_1 du_2.$$

Now, since  $K = \frac{eg - f^2}{EG - F^2}$ , we need to calculate the first and second fundamental form of the torus.

Let  $T$  be parametrized by

$$\phi(u_1, u_2) = ((a + r \cos u_1) \cos u_2, (a + r \cos u_1) \sin u_2, r \sin u_1).$$

Then

$$\begin{aligned} \Phi_1 &= \frac{\partial \phi}{\partial u_1} = (-r \sin u_1 \cos u_2, -r \sin u_1 \sin u_2, r \cos u_1) \\ \Phi_2 &= \frac{\partial \phi}{\partial u_2} = (-(a + r \cos u_1) \sin u_2, (a + r \cos u_1) \cos u_2, 0). \end{aligned}$$

Furthermore,

$$\begin{aligned} \frac{\partial^2 \phi}{\partial u_1^2} &= (-r \cos u_1 \cos u_2, -r \cos u_1 \sin u_2, -r \sin u_1), \\ \frac{\partial^2 \phi}{\partial u_2^2} &= (r \sin u_1 \sin u_2, -r \sin u_1 \cos u_2, 0) \end{aligned}$$

and

$$\frac{\partial^2 \phi}{\partial u_2^2} = (-(a + r \cos u_1) \cos u_2, -(a + r \cos u_1) \sin u_2, 0).$$

Now, to calculate  $N$ , we see that

$$\begin{aligned} N(u_1, u_2) &= \frac{\Phi_1 \times \Phi_2}{\|\Phi_1 \times \Phi_2\|} \\ &= \frac{(-r(a+r \cos u_1) \cos u_1 \cos u_2, -r(a+r \cos u_1) \cos u_1 \sin u_2, -r(a+r \cos u_1)(\sin u_1 \cos^2 u_2 - \sin u_1 \sin^2 u_2))}{\|\Phi_1 \times \Phi_2\|} \\ &= r(a+r \cos u_1) \frac{(-\cos u_1 \cos u_2, -\cos u_1 \sin u_2, -\sin u_1)}{\sqrt{r^2(a+r \cos u_1)^2(\cos^2 u_1 \cos^2 u_2 + \cos^2 u_1 \sin^2 u_2 + \sin^2 u_1)}} \\ &= (-\cos u_1 \cos u_2, -\cos u_1 \sin u_2, -\sin u_1). \end{aligned}$$

Now, we can calculate the coefficients of the first and second fundamental forms:

$$E = \langle \Phi_1, \Phi_1 \rangle = r^2 \cos^2 u_1 \cos^2 u_2 + r^2 \sin^2 u_1 \sin^2 u_2 + r^2 \cos^2 u_1 = r^2,$$

$$F = \langle \Phi_1, \Phi_2 \rangle = r(a+r \cos u_1) \sin u_1 \sin u_2 \cos u_2 - r(a+r \cos u_1) \sin u_1 \sin u_2 \cos u_2 = 0,$$

$$G = \langle \Phi_2, \Phi_2 \rangle = (a+r \cos u_1)^2 \sin^2 u_2 + (a+r \cos u_1)^2 \cos^2 u_2 = (a+r \cos u_1)^2,$$

$$e = \langle N, \frac{\partial^2 \phi}{\partial u_1^2} \rangle = r \cos^2 u_1 \cos^2 u_2 + r \cos^2 u_1 \sin^2 u_2 + r \sin^2 u_1 = r,$$

$$f = \langle N, \frac{\partial^2 \phi}{\partial u_1 \partial u_2} \rangle = -r \sin u_1 \sin u_2 \cos u_1 \cos u_2 + r \sin u_1 \sin u_2 \cos u_1 \cos u_2 = 0$$

and

$$g = \langle N, \frac{\partial^2 \phi}{\partial u_2^2} \rangle = (a+r \cos u_1) \cos u_1 \cos^2 u_2 + (a+r \cos u_1) \cos u_1 \sin^2 u_2 = (a+r \cos u_1) \cos u_1.$$

Therefore,

$$\begin{aligned} \int \int_T K d\sigma &= \int_0^{2\pi} \int_0^{2\pi} K \sqrt{EG - F^2} du_1 du_2 \\ &= \int_0^{2\pi} \int_0^{2\pi} \frac{eg - f^2}{\sqrt{EG - F^2}} du_1 du_2 \\ &= \int_0^{2\pi} \int_0^{2\pi} \frac{r(a+r \cos u_1) \cos u_1}{\sqrt{r^2(a+r \cos u_1)^2}} du_1 du_2 \\ &= \int_0^{2\pi} \int_0^{2\pi} \frac{r(a+r \cos u_1) \cos u_1}{r(a+r \cos u_1)} du_1 du_2 \\ &= \int_0^{2\pi} \int_0^{2\pi} \cos u_1 du_1 du_2 \\ &= 0, \end{aligned}$$

since  $\cos$  is  $2\pi$ -periodic.

Now, if we want to check this result with the Gauss-Bonnet theorem, we need only find a triangulation of the torus and count faces, edges and vertices in order to calculate the Euler-Poincare characteristic of the torus. Specifically, let us choose the triangulation given by the following drawing:

Then, since all vertices on this square are identified,  $V = 1$ . Certainly,  $F = 2$  and then, since opposite edges are identified, we see that  $E = 3$ . Then

$$\chi(S) = F - E + V = 2 - 3 + 1 = 0.$$

The Gauss-Bonnet Theorem (more specifically, Corollary 2), would suggest that

$$0 = 2\pi\chi(T) = \int \int_T K d\sigma,$$

which is exactly what we calculated above.



#### 4.5.4

Compute the Euler-Poincare characteristic of

(a) An ellipsoid.

**Answer:** Triangulate the ellipsoid as shown:

Then we simply count the faces, edges and vertices and calculate that the Euler-Poincare characteristic of the ellipse  $S$  is

$$\chi(S) = F - E + V = 4 - 4 + 2 = 2.$$



(b) The surface  $S = \{(x, y, z) \in \mathbb{R}^3; x^2 + y^{10} + z^6 = 1\}$ .

**Answer:** Define  $f : S \rightarrow \mathbb{R}^3$  by

$$(x, y, z) \mapsto (f_1(x, y, z), f_2(x, y, z), f_3(x, y, z)) = (x, y^5, z^3)$$

Then we see that the image of  $f$  is simply  $S^2$ : if  $(x, y, z) \in S$ , then

$$(f_1(x, y, z))^2 + (f_2(x, y, z))^2 + (f_3(x, y, z))^2 = (x)^2 + (y^5)^2 + (z^3)^2 = x^2 + y^{10} + z^6 = 1,$$

so  $f(x, y, z) \in S^2$ . Now, define  $g : S^2 \rightarrow S$  given by

$$(x, y, z) \mapsto (x, y^{1/5}, z^{1/3}).$$

Now, both  $f$  and  $g$  are certainly continuous, since their coordinate functions are (note that this is dependent on the fact that 1, 3 and 5 are all odd;  $g$  would not even be well-defined if one of these powers were even). Furthermore,

$$(g \circ f)(x, y, z) = g(f(x, y, z)) = g(x, y^5, z^3) = (x, y, z)$$

and

$$(f \circ g)(x, y, z) = f(g(x, y, z)) = f(x, y^{1/5}, z^{1/3}) = (x, y, z)$$

so, in fact,  $g = f^{-1}$ . Since  $f$  is continuous with continuous inverse,  $f$  is a homeomorphism of  $S$  with  $S^2$ . Now, since the Euler-Poincare characteristic is a topological invariant, we know that

$$\chi(S) = \chi(S^2) = 2.$$



4.5.6

Show that  $(0, 0)$  is an isolated singular point and compute the index at  $(0, 0)$  of the following vector fields in the plane:

(a)  $v = (x, y)$ .

**Answer:** Certainly  $(0, 0)$  is an isolated singular point of  $v$ , since  $v(p) = 0$  if and only if  $p = (0, 0)$ , meaning  $(0, 0)$  is the only singular point of  $v$ .

Now, restrict  $v$  to the closed curve given by  $\alpha(t) = (\cos t, \sin t)$  for  $t \in [0, 2\pi]$  and let  $\phi(t)$  denote the determination of the angle between  $v(t) := v(\alpha(t))$  and the  $x$ -axis. Then  $v(0) = (1, 0)$ , so  $\phi(0) = 0$ . Let  $a(t) = \cos t$ ,  $b(t) = \sin t$ . Then  $a$  and  $b$  fulfill the requirements of Lemma 1, Section 4.4. Hence,

$$\begin{aligned} \phi(t) &= \phi(0) + \int_0^t (ab' - ba') ds \\ &= \int_0^t (\cos^2 s + \sin^2 s) ds \\ &= \int_0^t ds \\ &= t. \end{aligned}$$

Therefore,

$$2\pi I = \phi(2\pi) - \phi(0) = 2\pi - 0 = 2\pi$$

so we can conclude that  $I = 1$ .



(b)  $v = (-x, y)$ .

**Answer:** Certainly  $(0, 0)$  is an isolated singular point of  $v$ , since  $v(p) = 0$  if and only if  $p = (0, 0)$ , meaning  $(0, 0)$  is the only singular point of  $v$ .

Now, restrict  $v$  to the closed curve given by  $\alpha(t) = (\cos t, \sin t)$  for  $t \in [0, 2\pi]$  and let  $\phi(t)$  denote the determination of the angle between  $v(t) := v(\alpha(t))$  and the  $x$ -axis. Then  $v(0) = (-1, 0)$ , so  $\phi(0) = \pi$ . Let  $a(t) = -\cos t$ ,  $b(t) = \sin t$ . Then  $a$  and  $b$  fulfill the requirements of Lemma 1, Section 4.4. Hence,

$$\begin{aligned} \phi(t) &= \phi(0) + \int_0^t (ab' - ba') ds \\ &= \pi + \int_0^t (-\cos^2 s - \sin^2 s) ds \\ &= \pi + \int_0^t -1 ds \\ &= \pi - t. \end{aligned}$$

Therefore,

$$2\pi I = \phi(2\pi) - \phi(0) = \pi - 2\pi - \pi = -2\pi$$

so we can conclude that  $I = -1$ .



(c)  $v = (x, -y)$ .

**Answer:** Certainly  $(0, 0)$  is an isolated singular point of  $v$ , since  $v(p) = 0$  if and only if  $p = (0, 0)$ , meaning  $(0, 0)$  is the only singular point of  $v$ .

Now, restrict  $v$  to the closed curve given by  $\alpha(t) = (\cos t, \sin t)$  for  $t \in [0, 2\pi]$  and let  $\phi(t)$  denote the determination of the angle between  $v(t) := v(\alpha(t))$  and the  $x$ -axis. Then  $v(0) = (1, 0)$ , so  $\phi(0) = 0$ . Let  $a(t) = \cos t$ ,  $b(t) = -\sin t$ . Then  $a$  and  $b$  fulfill the requirements of Lemma 1, Section 4.4. Hence,

$$\begin{aligned}\phi(t) &= \phi(0) + \int_0^t (ab' - ba') ds \\ &= \int_0^t (-\cos^2 s - \sin^2 s) ds \\ &= \int_0^t -1 ds \\ &= -t.\end{aligned}$$

Therefore,

$$2\pi I = \phi(2\pi) - \phi(0) = -2\pi + 0 = -2\pi$$

so we can conclude that  $I = -1$ .



(d)  $v = (x^2 - y^2, -2xy)$ .

**Answer:** Certainly  $(0, 0)$  is an isolated singular point of  $v$ , since  $v(x, y) = 0$  if and only if

$$x^2 - y^2 = 0 \quad \text{and} \quad -2xy = 0.$$

This in turn implies that either  $x = 0$  or  $y = 0$  by the second equation. From this point, we see that both  $x$  and  $y$  must be zero by the first equation. Hence,  $v(p) = 0$  if and only if  $p = (0, 0)$ , meaning  $(0, 0)$  is the only singular point of  $v$ .

Now, restrict  $v$  to the closed curve given by  $\alpha(t) = (\cos t, \sin t)$  for  $t \in [0, 2\pi]$  and let  $\phi(t)$  denote the determination of the angle between  $v(t) := v(\alpha(t))$  and the  $x$ -axis. Then

$$v(t) = (\cos^2 t - \sin^2 t, -2 \cos t \sin t) = (\cos 2t, -\sin 2t)$$

by the double-angle identities. Then  $v(0) = (1, 0)$ , so  $\phi(0) = 0$ . Let  $a(t) = \cos 2t$ ,  $b(t) = -\sin 2t$ . Then  $a$  and  $b$  fulfill the requirements of Lemma 1, Section 4.4. Hence,

$$\begin{aligned}\phi(t) &= \phi(0) + \int_0^t (ab' - ba') ds \\ &= \int_0^t (-2 \cos^2 2s - 2 \sin^2 2s) ds \\ &= \int_0^t -2 ds \\ &= -2t.\end{aligned}$$

Therefore,

$$2\pi I = \phi(2\pi) - \phi(0) = -4\pi + 0 = -4\pi$$

so we can conclude that  $I = -2$ .



(e)  $v = (x^3 - 3xy^2, y^3 - 3x^2y)$ .

**Answer:** Certainly  $(0, 0)$  is an isolated singular point of  $v$ , since  $v(x, y) = 0$  if and only if

$$x^3 - 3xy^2 = 0 \quad \text{and} \quad y^3 - 3x^2y = 0.$$

This in turn implies that

$$x^2 = 3y^2 \quad \text{and} \quad y^2 = 3x^2.$$

This, in turn, implies that

$$y^2 = 9y^2 \quad \text{and} \quad x^2 = 9x^2,$$

Hence,  $v(p) = 0$  if and only if  $p = (0, 0)$ , meaning  $(0, 0)$  is the only singular point of  $v$ .

Now, restrict  $v$  to the closed curve given by  $\alpha(t) = (\cos t, \sin t)$  for  $t \in [0, 2\pi]$  and let  $\phi(t)$  denote the determination of the angle between  $v(t) := v(\alpha(t))$  and the  $x$ -axis. Then

$$\begin{aligned} v(t) &= (\cos^3 t - 3 \cos t \sin^2 t, \sin^3 t - 3 \sin t \cos^2 t) \\ &= (\cos 2t, -\sin 2t) \end{aligned}$$

by the double-angle identities. Then  $v(0) = (1, 0)$ , so  $\phi(0) = 0$ . Let  $a(t) = \cos^3 t - 3 \cos t \sin^2 t$ ,  $b(t) = \sin^3 t - 3 \sin t \cos^2 t$ . Then

$$\begin{aligned} a^2 + b^2 &= (\cos^3 t - 3 \cos t \sin^2 t)^2 + (\sin^3 t - 3 \sin t \cos^2 t)^2 \\ &= \cos^6 t + 3 \cos^4 t \sin^2 t + 3 \cos^2 t \sin^4 t + \sin^6 t \\ &= (\cos^2 t + \sin^2 t)^3 \\ &= 1^3 \\ &= 1. \end{aligned}$$

Hence,  $a$  and  $b$  fulfill the requirements of Lemma 1, Section 4.4. Therefore,

$$\phi(t) = \phi(0) + \int_0^t (ab' - ba') ds$$

Now, calculating  $a'$  and  $b'$ , we see that

$$a'(t) = -3 \cos^2 t \sin t - 3(-\sin^3 t + 2 \cos^2 t \sin t) = -9 \cos^2 t \sin t + 3 \sin^3 t$$

and

$$b'(t) = 3 \sin^2 t \cos t - 2(\cos^3 t - 2 \sin^2 t \cos t) = 9 \sin^2 t \cos t - 3 \cos^3 t.$$

Thus,

$$\begin{aligned} \phi(t) &= \phi(0) + \int_0^t (ab' - ba') ds \\ &= \int_0^t ((\cos^3 s - 3 \cos s \sin^2 s)(-9 \cos^2 s \sin s + 3 \sin^3 s) \\ &\quad - (\sin^3 s - 3 \sin s \cos^2 s)(9 \sin^2 s \cos s - 3 \cos^3 s)) ds \\ &= \int_0^t (-3 \cos^6 s - 9 \cos^4 s \sin^2 s - 9 \cos^2 s \sin^4 s - 3 \sin^6 s) ds \\ &= \int_0^t -3(\cos^2 s + \sin^2 s)^3 ds \\ &= \int_0^t -3 ds \\ &= -3t. \end{aligned}$$

Therefore,

$$2\pi I = \phi(2\pi) - \phi(0) = -6\pi + 0 = -6\pi$$

so we can conclude that  $I = -3$ .



1

In class we defined the integral of a smooth function  $f : S \rightarrow \mathbb{R}$  over a bounded region  $R \subseteq S$  contained in a coordinate chart  $\phi : U \rightarrow V \subseteq S$  by

$$\int \int_{\phi^{-1}(R)} f(u_1, u_2) \sqrt{EG - F^2} du_1 du_2.$$

Show that this definition is independent of the coordinate chart which contains  $R$ .

*Proof.* Let  $\bar{\phi} : \bar{U} \rightarrow \bar{V} \subseteq S$  be another coordinate chart containing  $R$ . Let  $E, F, G$  be the coefficients of the first fundamental form associated with  $\phi$  and let  $\bar{E}, \bar{F}, \bar{G}$  be the coefficients of the first fundamental form associated with  $\bar{\phi}$ . Let  $h := \phi^{-1} \circ \bar{\phi}$  be the change-of-coordinate map and let  $\partial(u_1, u_2)/\partial(\bar{u}_1, \bar{u}_2)$  be the Jacobian of the change of coordinates. Then

$$\begin{aligned} & \int \int_{\bar{\phi}^{-1}(R)} f(\bar{u}_1, \bar{u}_2) \sqrt{\bar{E}\bar{G} - \bar{F}^2} d\bar{u}_1 d\bar{u}_2 \\ &= \int \int_{\phi^{-1}(R)} f(h(\bar{u}_1, \bar{u}_2)) \sqrt{EG - F^2} \left| \frac{\partial(u_1, u_2)}{\partial(\bar{u}_1, \bar{u}_2)} \right| d\bar{u}_1 d\bar{u}_2 \\ &= \int \int_{\bar{\phi}^{-1}(R)} f(u_1, u_2) \sqrt{EG - F^2} \left| \frac{\partial(u_1, u_2)}{\partial(\bar{u}_1, \bar{u}_2)} \right| d\bar{u}_1 d\bar{u}_2 \\ &= \int \int_{\phi^{-1}(R)} f(u_1, u_2) du_1 du_2. \end{aligned}$$

Hence, this integral is independent of the choice of coordinate chart.  $\square$