

GEOMETRY MID-TERM

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1. POLAR COORDINATES

\mathbb{R}^2 can be considered as a regular surface S by setting $S = \{(x, y, z) \in \mathbb{R}^3 : z = 0\}$. Polar coordinates on S are given as follows. Let $U = \{(\rho, \theta) : \rho > 0, 0 < \theta < 2\pi\}$ and let $\phi : U \rightarrow V \subset S$ be given by $\phi(\rho, \theta) = (\rho \cos \theta, \rho \sin \theta, 0)$, where $V = S \setminus \{(x, 0, 0) \in \mathbb{R}^3 : x \geq 0\}$.

(a) Compute the first fundamental form of S in polar coordinates.

Answer: First, we need to compute $\Phi_1 = \frac{\partial \phi}{\partial \rho}$ and $\Phi_2 = \frac{\partial \phi}{\partial \theta}$:

$$\Phi_1 = (\cos \theta, \sin \theta, 0)$$

$$\Phi_2 = (-\rho \sin \theta, \rho \cos \theta, 0).$$

Now, the coefficients of the first fundamental form are as follows:

$$\begin{aligned} E &= \langle \Phi_1, \Phi_1 \rangle = \cos^2 \theta + \sin^2 \theta = 1 \\ F &= \langle \Phi_1, \Phi_2 \rangle = -\rho \sin \theta \cos \theta + \rho \sin \theta \cos \theta = 0 \\ G &= \langle \Phi_2, \Phi_2 \rangle = \rho^2 \sin^2 \theta + \rho^2 \cos^2 \theta = \rho^2. \end{aligned}$$



(b) Compute the Christoffel symbols of S in polar coordinates. How does this compare with the computation in the usual coordinate system $\psi(x, y) = (x, y, 0)$?

Answer: In order to compute the Christoffel symbols, first we need to compute the partials of E , F and G :

$$\begin{aligned} E_\rho &= 0 & E_\theta &= 0 \\ F_\rho &= 0 & F_\theta &= 0 \\ G_\rho &= 2\rho & G_\theta &= 0 \end{aligned}$$

Now, plugging these values into the system of equations that lets us determine the Christoffel symbols, we see

$$\begin{aligned} 0 &= \frac{1}{2}E_\rho &= \langle \Phi_{11}, \Phi_1 \rangle &= \Gamma_{11}^1 E + \Gamma_{11}^2 F = \Gamma_{11}^1 \\ 0 &= F_\rho - \frac{1}{2}E_\theta &= \langle \Phi_{11}, \Phi_2 \rangle &= \Gamma_{11}^1 F + \Gamma_{11}^2 G = \Gamma_{11}^2 \rho^2 \\ 0 &= \frac{1}{2}E_\theta &= \langle \Phi_{12}, \Phi_1 \rangle &= \Gamma_{12}^1 E + \Gamma_{12}^2 F = \Gamma_{12}^1 \\ \rho &= \frac{1}{2}G_\rho &= \langle \Phi_{12}, \Phi_2 \rangle &= \Gamma_{12}^1 F + \Gamma_{12}^2 G = \Gamma_{12}^2 \rho^2 \\ -\rho &= F_\theta - \frac{1}{2}G_\rho &= \langle \Phi_{22}, \Phi_1 \rangle &= \Gamma_{22}^1 E + \Gamma_{22}^2 F = \Gamma_{22}^1 \\ 0 &= G_\theta &= \langle \Phi_{22}, \Phi_2 \rangle &= \Gamma_{22}^1 F + \Gamma_{22}^2 G = \Gamma_{22}^2 \rho^2. \end{aligned}$$

From this, then, we conclude that

$$\Gamma_{11}^1 = \Gamma_{11}^2 = \Gamma_{12}^1 = \Gamma_{22}^2 = 0$$

and

$$\Gamma_{12}^2 = \frac{1}{\rho}, \quad \Gamma_{22}^1 = -\rho.$$

These differ somewhat from the Christoffel symbols in the usual coordinate system, which are all zero, due to the fact that the partials of E , F and G in the usual coordinate system are all zero.



2. MAXIMAL INTEGRAL CURVES

On a previous HW we showed that on a compact manifold the maximal integral curves of a smooth vector field are defined for all time t . Show that on the open disk $D = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$ there exists a smooth vector field X for which the maximal integral curves are not defined for all time t . Hence, the compactness condition on our HW problem is necessary.

Example: Let $X_q = q$ for all $q \in D$. Specifically, the vector in $T_p D$ associated with the point $p = (x, y)$ is simply (x, y) . Now, let $p = (x_0, y_0) \in D$ and let $\alpha : I \rightarrow D$ be an integral curve through p of X . Then $\alpha(t) = (x(t), y(t))$ for $t \in I$,

$$\alpha(0) = (x_0, y_0)$$

and

$$(x'(t), y'(t)) = \alpha'(t) = X_{\alpha(t)} = (x(t), y(t)) = \alpha(t).$$

Hence, we see that $\alpha(t) = (x_0 e^t, y_0 e^t)$. Since the integral curves are unique, we see that curves of this form are the only integral curves on D ; specifically, any maximal integral curve must be of this form. Now, if $p = 0$, a maximal integral curve can be defined for all time t . However, this curve is constant, forever remaining fixed at the origin. On the other hand, if $p \neq 0$, then we see that, for all $t > t_0$ for some $t_0 \in \mathbb{R}$,

$$|(x_0 e^t, y_0 e^t)| > 1,$$

meaning α is not defined for these values of t . Since, as we argued above, all integral curves, including any maximal integral curves, must be of this form, we see that the maximal integral curves of X on D are not defined for all time t .



3. COVERING MAPS

Let \tilde{S} and S be two regular surfaces. A map $F : \tilde{S} \rightarrow S$ is said to be a **smooth covering map** if

(a) F is smooth.

(b) For each $p \in S$ there exists a neighborhood O of p such that $F^{-1}(O) = \cup_{\alpha \in J} V_\alpha$ is the *disjoint* union of open sets V_α such that for each $\alpha \in J$ we have $F : V_\alpha \rightarrow O$ is a diffeomorphism. Such a neighborhood O is said to be *evenly covered* by F .

Now suppose that $F_1 : S_1 \rightarrow S$ and $F_2 : S_2 \rightarrow S$ are smooth covering maps and $g : S_1 \rightarrow S_2$ is a homeomorphism such that $F_2 \circ g = F_1$ and $F_1 \circ g^{-1} = F_2$. Show that $g : S_1 \rightarrow S_2$ is a diffeomorphism.

Proof. Let $p \in S_1$ and let $q = F_1(p) = (F_2 \circ g)(p)$. Let O_1 be a neighborhood of q that is evenly covered by F_1 and let O_2 be a neighborhood of q that is evenly covered by F_2 . Then

$$F_1^{-1}(O_1) = \cup_{\alpha \in J} V_{1\alpha}$$

and

$$F_2^{-1}(O_2) = \cup_{\alpha \in J} V_{2\alpha}$$

where $F_1 : V_{1\alpha} \rightarrow O_1$ and $F_2 : V_{2\alpha} \rightarrow O_2$ are diffeomorphisms. Let $V_{1\beta}$ be the $V_{1\alpha}$ containing p and let $V_{2\beta}$ be the $V_{2\alpha}$ containing $g(p)$. Let

$$O = O_1 \cap O_2, \quad V_1 = V_{1\beta} \cap F_1^{-1}(O), \quad V_2 = V_{2\beta} \cap F_2^{-1}(O).$$

Then $F_1 : V_1 \rightarrow O$ and $F_2 : V_2 \rightarrow O$ are diffeomorphisms. Let

$$\phi_1 : U_1 \subset \mathbb{R}^2 \rightarrow W_1 \subset V_1$$

$$\phi_2 : U_2 \subset \mathbb{R}^2 \rightarrow W_2 \subset V_2$$

and

$$\phi : U \subset \mathbb{R}^2 \rightarrow W \subset O$$

be coordinate charts on p , $g(p)$ and q , respectively, such that $F_1(\phi_1(U_1)) \subseteq \phi(U)$ and $F_2(\phi_2(U_2)) \subseteq \phi(U)$. Then

$$H_1 := \phi^{-1} \circ F_1 \circ \phi_1 : U_1 \rightarrow U$$

and

$$H_2 := \phi^{-1} \circ F_2 \circ \phi_2 : U_2 \rightarrow U$$

are differentiable. Furthermore, if

$$\tilde{U} := H_1(U_1) \cap H_2(U_2),$$

$$\tilde{U}_1 := H_1^{-1}(\tilde{U})$$

and

$$\tilde{U}_2 := H_2^{-1}(\tilde{U})$$

then $H_1 : \tilde{U}_1 \rightarrow \tilde{U}$ and $H_2 : \tilde{U}_2 \rightarrow \tilde{U}$ are diffeomorphisms. Hence, $H_1^{-1} \circ H_2$ is differentiable, meaning that, when the domains of the F 's and ϕ 's are restricted appropriately,

$$\begin{aligned} H_1^{-1} \circ H_2 &= (\phi^{-1} \circ F_1 \circ \phi_1)^{-1}(\phi^{-1} \circ F_2 \circ \phi_2) \\ &= \phi_1^{-1} \circ F_1^{-1} \circ \phi \circ \phi^{-1} \circ F_2 \circ \phi_2 \\ &= \phi_1^{-1} \circ F_1^{-1} \circ F_2 \circ \phi_2 \\ &= \phi_1^{-1} \circ F_1^{-1} \circ F_1 \circ g^{-1} \circ \phi_2 \\ &= \phi_1^{-1} \circ g^{-1} \circ \phi_2 \end{aligned}$$

is differentiable. This says precisely that g^{-1} is differentiable on the appropriate neighborhood of $g(p)$.

A similar argument shows that g is differentiable on the corresponding neighborhood of p . Since our choice of p was arbitrary, we see that g is a differentiable homeomorphism with differentiable inverse at all points in S_1 : that is to say, g is a diffeomorphism. \square

4. NON-ORIENTABLE SURFACES

In do Carmo (and in class) Gaussian Curvature was developed only for orientable surfaces. Is this necessary? Please discuss.

Answer: No, it is not necessary for a surface to be orientable in order for us to determine the Gaussian Curvature at a point p on the surface. To see why this is the case, we first note that every surface covered by a single coordinate system is trivially orientable. Hence, any surface, even a non-orientable surface like the Mobius strip is locally orientable. Therefore, if we take our point p and consider some coordinate chart $\phi : U \subset \mathbb{R}^2 \rightarrow V$ on a neighborhood of p , then we see that we can assign an orientation to this neighborhood of V .

Given this orientation N , then, we can certainly define the Gauss map $N : V \rightarrow \mathbb{R}^3$ on this neighborhood V of p . In fact, since V is itself a regular surface, we can define the Gaussian curvature K , which is just the determinant of dN_p . Now, this determinant is not dependent on the choice of orientation for V (remember, V is an orientable surface), so this notion of Gaussian curvature at p is well-defined.



5. REGULAR N-MANIFOLDS

Let $M_2(\mathbb{R})$ be the set of all 2×2 matrices and $SL(2, \mathbb{R}) = \{A \in M_2(\mathbb{R}) : \det(A) = 1\}$ the set of all real 2×2 matrices with determinant 1.

(a) $M_2(\mathbb{R})$ can be identified with \mathbb{R}^4 via the map $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \mapsto (a, b, c, d)$.

Use this identification to show that $SL(2, \mathbb{R})$ is a regular 3-surface.

Proof. Let $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL(2, \mathbb{R})$, which is identified with $(a, b, c, d) \in \mathbb{R}^4$.

Since the zero matrix is not in $SL(2, \mathbb{R})$ we may assume, up to a change of axes, that $a \neq 0$. Then let $V = \{(x, y, z, w) \in \mathbb{R}^4 : x \neq 0\}$ and let $U = \{(x, y, z) \in \mathbb{R}^3 : x \neq 0\}$. Then V is certainly an open set of \mathbb{R}^4 and U is an open set of \mathbb{R}^3 . Now, let $\phi : U \rightarrow V$ be given by

$$(x, y, z) \mapsto \left(x, y, z, \frac{1 + yz}{x}\right).$$

We see that

$$\det \begin{bmatrix} x & y \\ z & \frac{1+yz}{x} \end{bmatrix} = x \frac{1+yz}{x} - yz = 1 + yz - yz = 1,$$

so the map ϕ is well-defined. Since $x \neq 0$, we see that ϕ is smooth since each coordinate function is. Also,

$$d\phi = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{-x(1+yz)}{x^2} & \frac{z}{x} & \frac{y}{x} \end{pmatrix}$$

The top minor of $d\phi$ has determinant 1 for all points in U , so $d\phi$ is injective. Hence, it only remains to show that ϕ is a homeomorphism. Certainly, since ϕ is smooth, ϕ is continuous, so we need only show that ϕ^{-1} exists and is continuous. If $A = (a, b, c, d) \in SL(2, \mathbb{R})$ (again, remember we may assume $a \neq 0$), then

$$1 = \det A = ad - bc \quad \Rightarrow \quad d = \frac{1 + bc}{a}$$

so $\phi^{-1}(a, b, c, d) = (a, b, c)$. Certainly this is a smooth function and, therefore continuous, so we see that ϕ is, indeed, a coordinate chart on $V \cap SL(2, \mathbb{R})$. Since our choice of (a, b, c, d) was arbitrary (up to change of axes), we conclude that $SL(2, \mathbb{R})$ is indeed a regular 3-surface. \square

(b) Show that the tangent space of $SL(2, \mathbb{R})$ at $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is given by

$$T_I(SL(2, \mathbb{R})) = \left\{ \left(I; \begin{bmatrix} a & b \\ c & d \end{bmatrix} \right) : a + d = 0 \right\}.$$

That is, $T_I(SL(2, \mathbb{R}))$ can be identified with the set of traceless 2×2 matrices over \mathbb{R} . This tangent space is an example of a Lie algebra.

Proof. By the theorem given in the statement of the problem, we know that

$$T_I(SL(2, \mathbb{R})) = d\phi_I(\mathbb{R}^3).$$

In part (a) above we gave the general form for $d\phi$, we need only plug in the coordinates of

$$\phi^{-1}(1, 0, 0, 1) = (1, 0, 0)$$

to that formula. Specifically, we see that, if $(x_1, x_2, x_3) \in \mathbb{R}^3$,

$$d\phi_I \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{-(1+0)}{1} & \frac{0}{1} & \frac{0}{1} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ -x_1 \end{pmatrix}.$$

Now, we identify this vector with the matrix

$$\begin{pmatrix} x_1 & x_2 \\ x_3 & -x_1 \end{pmatrix}.$$

We note that $x_1 + (-x_1) = 0$; when we vary (x_1, x_2, x_3) over all of \mathbb{R}^3 , we see that, indeed,

$$T_I(SL(2, \mathbb{R})) = d\phi_I(\mathbb{R}^3) = \left\{ \left(I; \begin{bmatrix} a & b \\ c & d \end{bmatrix} \right) : a + d = 0 \right\}.$$

□

6. VECTOR FIELDS AS DIFFERENTIAL OPERATORS

Let $X_p \in T_p S$ be a tangent vector and $f : U \subseteq S \rightarrow \mathbb{R}$ be a smooth function defined on an open neighborhood of p . Then

$$X_p f \equiv \frac{d}{dt}(f \circ \alpha)|_{t=0}$$

where $\alpha : (-\epsilon, \epsilon) \rightarrow S$ is a smooth curve such that $\alpha(0) = p$ and $\alpha'(0) = X_p$.

(a) Let X be a vector field on an open set U of S . Show that X is a smooth vector field if and only if for any function $f : U \rightarrow \mathbb{R}$ the function $q \mapsto X_q f$ is smooth.

Proof. Suppose X is a smooth vector field. This means that, if $\phi : W \subseteq \mathbb{R}^2 \rightarrow V \subseteq U$ is a coordinate chart on some point $p \in U$,

$$X_q = a(q)\Phi_{1q} + b(q)\Phi_{2q}$$

where $a, b : V \rightarrow \mathbb{R}$ are smooth maps. Let $\alpha : (-\epsilon, \epsilon) \rightarrow S$ be a smooth curve such that $\alpha(0) = q$ and $\alpha'(0) = X_q$. If $f : U \rightarrow \mathbb{R}$ is any smooth map, then, by the chain rule,

$$X_q f = \frac{d}{dt}(f \circ \alpha)|_{t=0} = df_{\alpha(0)}\alpha'(0) = df_{\alpha(0)}(a(q)\Phi_{1q} + b(q)\Phi_{2q}),$$

which is certainly a smooth function, since df, a, b, Φ_1 and Φ_2 are all smooth.

On the other hand, suppose X is a vector field and $q \mapsto X_q f$ is smooth. Then

$$X_q = a(q)\Phi_{1q} + b(q)\Phi_{2q}$$

where $\phi : W \subseteq \mathbb{R}^2 \rightarrow V \subseteq U$ is a coordinate chart on a neighborhood of $p \in U$. We want to show that $a, b : V \rightarrow \mathbb{R}$ are smooth. Let $\alpha : (-\epsilon, \epsilon) \rightarrow U$ be a smooth curve such that $\alpha(0) = q$ and $\alpha'(0) = X_q$. Then,

$$X_q f = \frac{d}{dt}(f \circ \alpha)|_{t=0} = df_{\alpha(0)}\alpha'(0) = df_{\alpha(0)}(a(q)\Phi_{1q} + b(q)\Phi_{2q}) = df_q a(q)\Phi_{1q} + df_q b(q)\Phi_{2q}$$

since df is a linear map. Since $q \mapsto X_q f$ is smooth, it follows that a and b are smooth, meaning that X is smooth. □

(b) Let $\lambda \in \mathbb{R}$ and $f, g : U \rightarrow \mathbb{R}$ be smooth functions. Show that for any $p \in S$ we have

$$X_p(\lambda f + \mu g) = \lambda X_p f + \mu X_p g$$

and

$$X_p(fg) = X_p(f)g(p) + f(p)X_p(g).$$

Proof. Let $\alpha : (-\epsilon, \epsilon) \rightarrow S$ be a smooth curve such that $\alpha(0) = p$ and $\alpha'(0) = X_p$. Since $\lambda f + \mu g$ is smooth, we can use the chain rule to see that

$$\begin{aligned} X_p(\lambda f + \mu g) &= \frac{d}{dt}((\lambda f + \mu g) \circ \alpha)|_{t=0} \\ &= d(\lambda f + \mu g)_{\alpha(0)}\alpha'(0) \\ &= (d(\lambda f)_{\alpha(0)} + d(\mu g)_{\alpha(0)})\alpha'(0) \\ &= (\lambda df_{\alpha(0)} + \mu dg_{\alpha(0)})\alpha'(0) \\ &= \lambda df_{\alpha(0)}\alpha'(0) + \mu dg_{\alpha(0)}\alpha'(0) \\ &= \lambda \frac{d}{dt}(f \circ \alpha)|_{t=0} + \mu \frac{d}{dt}(g \circ \alpha)|_{t=0} \\ &= \lambda X_p f + \mu X_p g. \end{aligned}$$

On the other hand, using the product rule for differentials and the chain rule,

$$\begin{aligned} X_p(fg) &= \frac{d}{dt}(fg \circ \alpha)|_{t=0} \\ &= d(fg)_{\alpha(0)}\alpha'(0) \\ &= (df_{\alpha(0)}g(\alpha(0)) + f(\alpha(0))dg_{\alpha(0)})\alpha'(0) \\ &= df_{\alpha(0)}g(p)\alpha'(0) + f(p)dg_{\alpha(0)}\alpha'(0) \\ &= df_{\alpha(0)}\alpha'(0)g(p) + f(p)dg_{\alpha(0)}\alpha'(0) \\ &= \frac{d}{dt}(f \circ \alpha)|_{t=0}g(p) + f(p)\frac{d}{dt}(g \circ \alpha)|_{t=0} \\ &= X_p(f)g(p) + f(p)X_p(g). \end{aligned}$$

□

(c) Show that for any smooth function $f : W \subseteq S \rightarrow \mathbb{R}$ on a neighborhood of $p \in S$ and any tangent vector $X_p \in T_p S$ we have

$$df_p(X_p) = X_p f.$$

Proof. Let α be a smooth curve such that $\alpha(0) = p$ and $\alpha'(0) = X_p$. Now, since f is smooth we may use the chain to see that

$$X_p f = \frac{d}{dt}(f \circ \alpha)|_{t=0} = df_{\alpha(0)} \cdot \alpha'(0) = df_p X_p.$$

Hence, we conclude that

$$df_p(X_p) = X_p f.$$

□

(d) Let X, Y, Z be smooth vector fields on S . Then the function $q \mapsto \langle Y_q, Z_q \rangle$ is a smooth function on S . Show that for any $p \in S$ we have

$$X_p \langle Y, Z \rangle = \langle \nabla_X Y(p), Z_p \rangle + \langle Y_p, \nabla_X Z(p) \rangle.$$

Proof. Again, let α be a smooth curve such that $\alpha(0) = p$ and $\alpha'(0) = X_p$. Using the result proved in class that, if X_1, X_2 are vector fields along α , then

$$\frac{d}{dt} \langle X_1(t), X_2(t) \rangle = \left\langle \frac{DX_2}{dt}, X_1 \right\rangle + \left\langle X_1, \frac{DX_1}{dt} \right\rangle,$$

we see that

$$\begin{aligned}
 X_p \langle Y, Z \rangle &= \frac{d}{dt} (\langle Y, Z \rangle \circ \alpha) |_{t=0} \\
 &= \frac{d}{dt} (\langle Y(t), Z(t) \rangle) |_{t=0} \\
 &= \left\langle \frac{DY}{dt}, Z_p \right\rangle + \left\langle Y_p, \frac{DZ}{dt} \right\rangle \\
 &= \langle \nabla_X Y(p), Z_p \rangle + \langle Y_p, \nabla_X Z(p) \rangle.
 \end{aligned}$$

□

7. VECTOR FIELDS & COORDINATE CHARTS

Let X be a smooth vector field on S and suppose $p \in S$ is such that $X_p \neq 0$. Show that there exists a coordinate chart $\phi : U \rightarrow V$ containing p such that $\Phi_{1q} = X_q$ for all $q \in V$.

Proof. We show this in two parts. First we prove the result on an open set contained in \mathbb{R}^2 , then extend that result to any regular surface.

Lemma 7.1. *Let X be a vector field on an open set $W \subset \mathbb{R}^2$ and let $p \in U$ such that $X_p \neq 0$. Then there exists a coordinate chart $\phi : U \rightarrow V \subseteq W$ on a neighborhood of p such that $\Phi_{1q} = X_q$ for all $q \in V$.*

Proof. Choose a coordinate system on \mathbb{R}^2 such that $p = (0, 0)$ and X_p is in the direction of the x axis. Let $\alpha : \tilde{V} \times I \rightarrow W$ be the local flow at p , $\tilde{V} \subset W$, $t \in I$ and let $\tilde{\phi}$ be the restriction of α to the rectangle

$$(\tilde{V} \times I) \cap \{(x, y, t) \in \mathbb{R}^3; x = 0\}.$$

We can identify $\{(x, y, t) \in \mathbb{R}^3 : x = 0\}$ with \mathbb{R}^2 in such a way that the t direction is identified with the u_1 direction and the y direction is identified with the u_2 direction. The above rectangle, then, is a subset of the plane and we see that $\tilde{\phi}$ is a map from a subset of \mathbb{R}^2 into W . By the definition of local flow, $d\tilde{\phi}_p$ maps the unit vector in the u_1 direction into X and maps the unit vector of the u_2 axis into itself. Therefore, $d\tilde{\phi}_p$ is nonsingular. By the inverse function theorem, then, it follows that there exists a neighborhood $V \subset W$ of p where $\tilde{\phi}^{-1}$ is defined and differentiable. Now, let $U = \tilde{\phi}^{-1}(V)$ and let ϕ be the restriction of $\tilde{\phi}$ to U . Since ϕ is a differentiable function with differentiable inverse, bijective and with nonsingular differential, we see that $\phi : U \rightarrow V$ is a coordinate chart on a neighborhood of p . It only remains to show that $\Phi_{1q} = X_q$ for all $q \in V$.

However, this follows immediately from how we have defined ϕ . Since the u_1 direction in U is identified with the t direction in the original rectangle lying in the yt -plane,

$$\frac{\partial \phi}{\partial u_1}(u_1, u_2) = \frac{\partial \alpha}{\partial t}(q, t) = X_{\alpha(q,t)}.$$

Hence, if $q \in V$ and $(u_0, v_0) \in U$ such that $\phi(u_0, v_0) = q$ (that is to say, u_0 and v_0 are identified with $t_0 \in I$ and q_0 is the unique point on the y -axis

such that the trajectory through q_0 passes through q), then

$$\phi_{1q} = \frac{\partial \phi}{\partial u_1}(u_0, v_0) = \frac{\partial \alpha}{\partial t}(q_0, t_0) = X_{\alpha(q_0, t_0)} = X_q.$$

□

Now, we turn to the general case.

Let S , X and p be as stated in the problem and let $\psi : U' \rightarrow V' \subset S$ be a coordinate chart on a neighborhood of p . Since X is smooth, we know that in the coordinate system,

$$X_q = a(q)\Psi_{1q} + b(q)\Psi_{2q}$$

where $a, b : V' \rightarrow \mathbb{R}$ are smooth. Consider the vector field $\tilde{X} = \tilde{a}e_1 + \tilde{b}e_2$ where

$$\tilde{a} = a \circ \psi, \tilde{b} = b \circ \psi : U' \rightarrow \mathbb{R}.$$

Then, as in the proof of the local first integral lemma for surfaces that we did in class, $X = d\psi(\tilde{X})$. Now, by the above lemma, there exists a coordinate chart $\tilde{\phi} : U \rightarrow \tilde{V} \subset U'$ such that $\tilde{\Phi}_{1v} = \tilde{X}_v$ for all $v \in \tilde{V}$. Define

$$\phi := \psi \circ \tilde{\phi} : U \rightarrow V = \psi(\tilde{V}).$$

Then, by the chain rule,

$$d\phi = d\psi \cdot d\tilde{\phi}.$$

Specifically,

$$\Phi_{1q} = d\psi_v(\tilde{\Phi}_{1v}) = d\psi_v(\tilde{X}_v) = X_q$$

where $v = \psi^{-1}(q)$.

□