ALGEBRA HW 7

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1

Prove that $\mathbb{R}[\mathbb{Z}/2\mathbb{Z}]$ is isomorphic to the product ring $\mathbb{R} \times \mathbb{R}$.

Proof. Define the map $\phi : \mathbb{R}[\mathbb{Z}/2\mathbb{Z}] \to \mathbb{R} \times \mathbb{R}$ by

$$\phi(a[0] + b[1]) = (a, b).$$

Then

$$ker(\phi) = \{a[0] + b[1] | a = 0, b = 0\} = \{0\},$$

so ϕ is injective. Also, for any $(c, d) \in \mathbb{R} \times \mathbb{R}$,

$$\phi(c[0] + d[1]) = (c, d),$$

so ϕ is surjective. Finally, for $a[0] + b[1], c[0] + d[1] \in \mathbb{R}[\mathbb{Z}/2\mathbb{Z}]$,

$$\begin{aligned} \phi((a[0] + b[1])(c[0] + d[1])) &= \phi((ac + bd)[0] + (ad + bc)[1]) \\ &= (ac + bd, ad + bc) \\ &= (a, b) \cdot (c, d) \\ &= \phi(a[0] + b[1]) \cdot \phi(c[0] + d[1]). \end{aligned}$$

Hence, ϕ is a homomorphism and, since it is bijective, an isomorphism. □

2

Show that for any non-trivial finite group G, there exist non-zero elements $u, v \in \mathbb{Z}[G]$ such that $u \cdot v = 0$.

Proof. Let p be a prime factor of $|G|$. Then there exists $g \in G$ such that $|g| = p$. If $p = 2$, then $(1[g] + 1[e]), (1[g] - 1[e])$ are non-zero elements of $\mathbb{Z}[G]$ and

$$(1[g]+1[e])(1[g]-1[e]) = 1[g]^2 - 1[e][g] + 1[e][g] - 1[e]^2 = 1[g]^2 - 1[e]^2 = 1[e] - 1[e] = 0.$$

If $p > 2$, then $(1[g]^{p-1} + 1[g]^2), (1[g]^{p-1} - 1[g]^2)$ are non-zero elements of $\mathbb{Z}[G]$ and

$$(1[g]^{p-2} + 1[g]^2)(1[g]^{p-2} - 1[g]^2) = 1[g]^{p^2-4p+4} - 1[g]^4 = 1[g]^4 - 1[g]^4 = 0.$$

□

3

Let $\zeta = \frac{-1+\sqrt{-3}}{2}$, a third root of unity in $\mathbb{C}$. Denote by $\mathbb{Q}(\sqrt{-3})$ the subring of $\mathbb{C}$ consisting of all elements of the form $a + b\sqrt{-3}$, with $a, b \in \mathbb{Q}$.

(i) Show that $\mathbb{Q}(\sqrt{-3})$ is a field.

Proof. Let $a + b\sqrt{-3}, c + d\sqrt{-3} \in \mathbb{Q}(\sqrt{-3})$. Then

$$(a + b\sqrt{-3}) + (c + d\sqrt{-3}) = (a + c) + (b + d)\sqrt{-3} \in \mathbb{Q}(\sqrt{-3}),$$
$$(a + b\sqrt{-3})(c + d\sqrt{-3}) = (ac - 3bd) + (ad + bc)\sqrt{-3} \in \mathbb{Q}(\sqrt{-3}),$$

so $\mathbb{Q}(\sqrt{-3})$ is closed under addition and multiplication. Furthermore, addition and multiplication are associative, since they are in $\mathbb{C}$, and the distributive law holds for the same reason. Also, $0 = 0+0\sqrt{-3} \in \mathbb{Q}(\sqrt{-3})$ and $1 = 1+0\sqrt{-3} \in \mathbb{Q}(\sqrt{-3})$. As for inverses, if $0 \neq a+b\sqrt{-3} \in \mathbb{Q}(\sqrt{-3})$, then

$$(a + b\sqrt{-3}) + (-a - b\sqrt{-3}) = 0$$

and

$$\begin{aligned}&\left(a + b\sqrt{-3}\right)\left(\tfrac{a}{a^2+3b^2} - \tfrac{b}{a^2+3b^2}\sqrt{-3}\right)\\ &= \left(\tfrac{a^2}{a^2+3b^2} + \tfrac{3b^2}{a^2+3b^2}\right) + \left(\tfrac{ab}{a^2+3b^2} - \tfrac{ab}{a^2+3b^2}\right)\sqrt{-3}\\ &= \tfrac{a^2+3b^2}{a^2+3b^2} + 0\sqrt{-3} = 1,\end{aligned}$$

so $\mathbb{Q}(\sqrt{-3})$ contains additive and multiplicative inverses for each element. Finally, we need only note that addition and multiplication are commutative to conclude that $\mathbb{Q}(\sqrt{-3})$ is a field. □

(ii) Find all ring homomorphisms from $\mathbb{Q}[\mathbb{Z}/3\mathbb{Z}]$ to $\mathbb{Q}(\sqrt{-3})$.

(iii) Find all ring homomorphisms from $\mathbb{Q}(\sqrt{-3})$ to $\mathbb{Q}[\mathbb{Z}/3\mathbb{Z}]$.

(iv) Find all ideals of $\mathbb{Q}[\mathbb{Z}/3\mathbb{Z}]$. Which ones among them are prime ideals?

4

Recall that $\mathbb{H}$ is the ring of Hamiltonian quaternions. Let Q_8 be the quaternion group, a subring of $\mathbb{H}^\times$ with elements $\pm 1, \pm i, \pm j, \pm k$. Let $h : \mathbb{R}[Q_8] \to \mathbb{H}$ be the ring homomorphism induced by the embedding $Q_8 \hookrightarrow \mathbb{H}^\times$.

(i) Show that $\mathrm{Ker}(h)$ consists of all $\mathbb{R}$-linear combinations of the elements

$$[1] + [-1], [i] + [-i], [j] + [-j], [k] + [-k]$$

of $\mathbb{R}[Q_8]$.

Proof. Let $a = a_1[1]+a_2[-1]+a_3[i]+a_4[-i]+a_5[j]+a_6[-j]+a_7[k]+a_8[-k] \in \mathbb{R}[Q_8]$. Then, $a \in \mathrm{Ker}(h)$ iff

$$0 = h(a) = (a_1 - a_2) + (a_3 - a_4)i + (a_5 - a_6)j + (a_7 - a_8)k,$$

which holds if and only if

$$\begin{aligned}a_1 &= a_2\\ a_3 &= a_4\\ a_5 &= a_6\\ a_7 &= a_8.\end{aligned}$$

In other words, $a \in \operatorname{Ker}(h) \Leftrightarrow$

$$a = a_1([1]+[-1]) + a_3([i]+[-i]) + a_5([j]+[-j]) + a_7([k]+[-k]),$$

a linear combination of the terms listed above. Therefore, $\operatorname{Ker}(h)$ consists of all such $\mathbb{R}$-linear combinations. ☐

(ii) Prove that $\operatorname{Ker}(h)$, as a ring, is not isomorphic to $\mathbb{H}$, nor to $M_2(\mathbb{R})$.

Proof. Let $a, b \in \operatorname{Ker}(h)$. Then

$$a = a_1([1]+[-1]) + a_2([i]+[-i]) + a_3([j]+[-j]) + a_4([k]+[-k])$$

$$b = b_1([1]+[-1]) + b_2([i]+[-i]) + b_3([j]+[-j]) + b_4([k]+[-k])$$

For some $a_i, b_i \in \mathbb{R}$, $i = \{1, \dots 4\}$. Then

$$\begin{aligned} ab = \quad & (2a_1b_1 + 2a_2b_2 + 2a_3b_3 + 2a_4b_4)[1] \\ & +(2a_1b_1 + 2a_2b_2 + 2a_3b_3 + 2a_4b_4)[-1] \\ & +(2a_1b_2 + 2a_2b_1 + 2a_3b_4 + 2a_4b_3)[i] \\ & +(2a_1b_2 + 2a_2b_1 + 2a_3b_4 + 2a_4b_3)[-i] \\ & +(2a_1b_3 + 2a_3b_1 + 2a_2b_4 + 2a_4b_2)[j] \\ & +(2a_1b_3 + 2a_3b_1 + 2a_2b_4 + 2a_4b_2)[-j] \\ & +(2a_1b_4 + 2a_4b_1 + 2a_2b_3 + 2a_3b_2)[k] \\ & +(2a_1b_4 + 2a_4b_1 + 2a_2b_3 + 2a_3b_2)[-k] \\ = \quad & ba. \end{aligned}$$

Hence, $\operatorname{Ker}(h)$ is a commutative ring, so it cannot be isomorphic to either $\mathbb{H}$ or $M_2(\mathbb{R})$, since both of these rings are noncommutative. ☐

5

Let I_0 be the left ideal of $M_2(\mathbb{R})$ consisting of all 2×2 matrices with entries in $\mathbb{R}$ whose second column is zero. Prove that every non-trivial proper left ideal of $M_2(\mathbb{R})$ has the form $I_0 \cdot A$ for some $A \in GL_2(\mathbb{R})$.

Proof. Let $B = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in J$, a proper non-trivial ideal of $M_2(\mathbb{R})$. J cannot contain any invertible matrices (if it did, it would necessarily contain the identity and therefore not be proper), so we know that $\det(B) = 0$. Specifically,

$$0 = \det B = ad - bc.$$

This means $ad = bc$. If we let $m = \frac{d}{b}$, it is immediately apparent that $c = ma$, $d = mb$. Therefore,

$$B = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ ma & mb \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ m & 0 \end{pmatrix} \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}.$$

☐

First, note that, for any $K = \begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix} \in I_o$

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