

ALGEBRA HW 2

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1

Let χ and χ' be the characters of two representations. Prove the formulas:

$$(\chi + \chi')_{\sigma}^2 = \chi_{\sigma}^2 + \chi'_{\sigma}{}^2 + \chi\chi',$$

$$(\chi + \chi')_{\alpha}^2 = \chi_{\alpha}^2 + \chi'_{\alpha}{}^2 + \chi\chi'.$$

Proof. We simply use the the fact that

$$\chi_{\sigma}^2(g) = \frac{1}{2}(\chi^2(g) + \chi(g^2))$$

and

$$\chi_{\alpha}^2(g) = \frac{1}{2}(\chi^2(g) + \chi(g^2))$$

and compute.

$$\begin{aligned}(\chi + \chi')_{\sigma}^2(g) &= \frac{1}{2}((\chi + \chi')^2(g) + (\chi + \chi')(g^2)) \\ &= \frac{1}{2}(\chi^2(g) + 2\chi\chi'(g) + \chi'^2(g) + \chi(g^2) + \chi'(g^2)) \\ &= \frac{1}{2}(\chi^2(g) + \chi(g^2)) + \frac{1}{2}(\chi'^2(g) + \chi'(g^2)) + \chi\chi'(g) \\ &= \chi_{\sigma}^2 + \chi'_{\sigma}{}^2 + \chi\chi'.\end{aligned}$$

Also

$$\begin{aligned}(\chi + \chi')_{\alpha}^2(g) &= \frac{1}{2}((\chi + \chi')^2(g) - (\chi + \chi')(g^2)) \\ &= \frac{1}{2}(\chi^2(g) + 2\chi\chi'(g) + \chi'^2(g) - \chi(g^2) - \chi'(g^2)) \\ &= \frac{1}{2}(\chi^2(g) - \chi(g^2)) + \frac{1}{2}(\chi'^2(g) - \chi'(g^2)) + \chi\chi'(g) \\ &= \chi_{\alpha}^2 + \chi'_{\alpha}{}^2 + \chi\chi'.\end{aligned}$$

□

2

Let X be a finite set on which G acts, let ρ be the corresponding permutation representation, and let χ_X be the character of ρ . Let $s \in G$; show that $\chi_X(s)$ is the number of elements of X fixed by s .

Proof. Now, by definition,

$$\chi_X(s) = \text{Tr}(\rho_s).$$

Now, considering ρ_s in matrix form, as $(\lambda_{xy})_{x,y \in X}$. Then

$$\chi_X(s) = \sum_{x \in X} \lambda_{xx}.$$

Now, recalling that $\rho(e_x) = e_{sx}$, then $\lambda_{xx} = 1$ precisely when $sx = x$. On the other hand, if $sx \neq x$, then $\lambda_{xx} = 0$. Hence,

$$\chi_X(s) = \sum_{x \in X} \lambda_{xx} = n_s,$$

where n_s is the number of elements of X fixed by s . \square

3

Let $\rho : G \rightarrow GL(V)$ be a linear representation with character χ and let V' be the dual of V , i.e., the space of linear forms on V . For $x \in V$, $x' \in V'$ let $\langle x, x' \rangle$ denote the value of the linear form x' at x . Show that there exists a unique linear representation $\rho' : G \rightarrow GL(V')$ such that

$$\langle \rho_s x, \rho'_s x' \rangle = \langle x, x' \rangle \text{ for } s \in G, x \in V, x' \in V'.$$

This is called the *contragredient* (or *dual*) representation of ρ ; its character is χ^* .

Proof. Let $\{e_g\}_{g \in G}$ be the basis for V . This basis induces a basis $\{e'_g\}_{g \in G}$ on V' , and so, if $x \in V$, x induces a unique element $x' \in V'$. If we define the action of ρ'_s on each e'_g , then this will completely determine ρ'_s . Hence, define ρ'_s as follows:

$$\rho'_s e'_g := e'_g \circ \rho_{s^{-1}}$$

for each $s \in G$ and extend linearly. Then, if $s \in G$,

$$\langle \rho_s e_g, \rho'_s e'_g \rangle = \langle \rho_s e_g, e'_g \circ \rho_{s^{-1}} \rangle = \langle \rho_s e_g, e'_g \circ \rho_s^{-1} \rangle = \langle e_g, e'_g \rangle$$

for each $g \in G$, and so we see that

$$\langle \rho_s x, \rho'_s x' \rangle = \langle x, x' \rangle$$

for all $x \in V$.

To show uniqueness, suppose there were another representation $\phi : G \rightarrow GL(V')$ such that

$$\langle \rho_s x, \phi_s x' \rangle = \langle x, x' \rangle = \langle \rho_s x, \rho'_s x' \rangle.$$

Then, subtracting the right from the left, we see that

$$0 = \langle \rho_s x, \phi_s x' \rangle - \langle \rho_s x, \rho'_s x' \rangle = \langle \rho_s x, \phi_s x' - \rho'_s x' \rangle = \langle \rho_s x, (\phi_s - \rho'_s) x' \rangle$$

for all $x \in V$ since ρ_s , ϕ_s and ρ'_s are linear. Now, since $\rho_s \in GL(V)$, $\rho_s x$ spans all of V as we let x vary. Hence

$$\langle \rho_s x, (\phi_s - \rho'_s) x' \rangle = 0$$

implies that $(\phi_s - \rho'_s) x' = 0$. Since this is true for all x , we see that it must be the case that $\phi_s - \rho'_s = 0$, or

$$\phi_s = \rho'_s.$$

Hence, ρ' is unique.

Now, as we've constructed it, ρ_s makes the following diagram commute for all $s \in G$:

$$\begin{array}{ccc} V & \xrightarrow{\rho_{s^{-1}}} & V \\ \rho'_s x' \downarrow & & \downarrow x' \\ \mathbb{C} & \longrightarrow & \mathbb{C} \end{array}$$

In other words,

$$\rho_{s^{-1}}(v) = x'^{-1} \circ \rho'_s \circ x'(v)$$

so

$$\chi_\rho(s^{-1}) = \text{Tr}(\rho_{s^{-1}}) = \text{Tr}(\rho'_s) = \chi_{\rho'}(s).$$

Now, to complete the proof, we simply note that

$$\chi_{\rho'}(s) = \chi_\rho(s^{-1}) = \chi_\rho^*(s).$$

□

4

Let $\rho_1 : G \rightarrow GL(V_1)$ and $\rho_2 : G \rightarrow GL(V_2)$ be two linear representations with characters χ_1 and χ_2 . Let $W = \text{Hom}(V_1, V_2)$, the vector space of linear mappings $f : V_1 \rightarrow V_2$. For $s \in G$ and $f \in W$ let $\rho_s f = \rho_{2,s} \circ f \circ \rho_{1,s}^{-1}$; so $\rho_s f \in W$. Show that this defines a linear representation $\rho : G \rightarrow GL(W)$, and that its character is $\chi_1^* \cdot \chi_2$.

Proof. We start out showing that ρ is a homomorphism, i.e. that for $g, h \in G$, $\rho_{gh} f = (\rho_g \circ \rho_h) f$. Note that, since ρ_1, ρ_2 are linear representations,

$$\rho_{2,gh} = \rho_{2,g} \circ \rho_{2,h}$$

and

$$\rho_{1,gh}^{-1} = \rho_{1,h}^{-1} \circ \rho_{1,g}^{-1}.$$

Using this data, we make the following calculation:

$$\rho_{gh} f = \rho_{2,gh} \circ f \circ \rho_{1,gh}^{-1} = \rho_{2,g} \circ \rho_{2,h} \circ f \circ \rho_{1,h}^{-1} \circ \rho_{1,g}^{-1} = \rho_g \circ (\rho_{2,h} \circ f \circ \rho_{1,h}^{-1}) = (\rho_g \circ \rho_h) f.$$

Hence, ρ is indeed a representation.

Now, we simply recall that

$$\text{Hom}(V_1, V_2) \simeq V_1' \otimes V_2.$$

Hence, $\rho = \tilde{\rho} \otimes \hat{\rho}$ where $\tilde{\rho} : G \rightarrow GL(V_1')$ and $\hat{\rho} : G \rightarrow GL(V_2)$ are uniquely determined representations. The only such $\tilde{\rho}$ and $\hat{\rho}$ are ρ_1' and ρ_2 , respectively, so $\rho = \rho_1' \otimes \rho_2$. In 3 above, we showed that the character of ρ_1' is χ_1^* , so we see that

$$\chi = \chi_1^* \cdot \chi_2.$$

□

5

Let ρ be a linear representation with character χ . Show that the number of times that ρ contains the unit representation is equal to $(\chi|1) = (1/g) \sum_{s \in G} \chi(s)$ where $g = |G|$.

Proof. Now, if $\rho' : G \rightarrow GL(W)$ is the trivial representation, then $\chi_{\rho'}(s) = 1$ for all $s \in G$. Suppose V is the vector space associated with the representation ρ . Now, we have seen that V is a direct sum of irreducible representations

$$V = W_1 \oplus \cdots \oplus W_k.$$

Recall that W is the trivial representation, which is irreducible. Then, by Theorem 4, the number of W_i isomorphic to W is equal to $(\chi|\chi_{\rho'})$ (recalling that ρ' is the trivial representation). Now

$$(\chi|1) = (\chi|\chi_{\rho'}) = \frac{1}{g} \sum_{s \in G} \chi(s) \chi_{\rho'}^*(s) = \frac{1}{g} \sum_{s \in G} \chi(s) \cdot 1 = \frac{1}{g} \sum_{s \in G} \chi(s).$$

□

6

Let X be a finite set on which G acts, let ρ be the corresponding permutation representation and let χ be its character.

(a) The set Gx of images under G of an element $x \in X$ is called an *orbit*. Let c be the number of distinct orbits. Show that c is equal to the number of times that ρ contains the unit representation 1; deduce from this that $(\chi|1) = c$. In particular, if G is transitive (i.e., if $c = 1$), ρ can be decomposed into $1 \oplus \theta$ and θ does not contain the unit representation. If ψ is the character of θ , we have $\chi = 1 + \psi$ and $(\psi|1) = 0$.

Proof. Let W denote the permutation representation and let $\{e_x\}_{x \in X}$ be its basis. Let $x \in X$ and consider the space X_y spanned by the vector

$$y = \sum_{z \in Gx} e_z.$$

Then

$$\rho_s y = \sum_{z \in Gx} \rho_s e_z = \sum_{z \in Gx} e_{sz},$$

which is just a re-ordering of y , so $\rho_s y = y$ for all $s \in G$. Hence, X_y is invariant under the action of G , so X_y is a subrepresentation of W isomorphic to the trivial representation. Since our choice of x was arbitrary, we see that we can obtain such a copy of the trivial representation for each of the c orbits of X . Since no other subspace of W is fixed by the action of G , we see that c is precisely the number of copies of the trivial representation in ρ . By what we showed in problem 5, then, $(\chi|1) = c$. If $c = 1$, then it is certainly true that ρ can be decomposed into $1 \oplus \theta$, and it is clear that θ does not contain the trivial representation, else there would be at least

two copies of it in ρ . Since the character of a direct sum is the sum of the characters, it follows that

$$\chi = 1 + \psi.$$

The fact that $(\psi|1) = 0$ is a result of the fact that ψ does not contain the trivial representation (again, by our work in problem 5 above). \square

(b) Let G act on the product $X \times X$ of X by itself by means of the formula $s(x, y) = (sx, sy)$. Show that the character of the corresponding permutation representation is equal to χ^2 .

Proof. By the definition of the permutation representation, we know that it has

$$\{e_{(x,y)}\}_{(x,y) \in X \times X}$$

as a basis. Let W be as above and let ρ' denote the permutation representation of $X \times X$ with corresponding vector space W' . Then we see that there is a bijection between W' and $W \otimes W$ given by

$$e_{(x,y)} \leftrightarrow e_x \otimes e_y$$

and extended linearly. Furthermore,

$$(\rho_s \otimes \rho_s)(e_x \otimes e_y) = e_{sx} \otimes e_{sy} \leftrightarrow e_{(sx,sy)} = \rho'_s e_{(x,y)},$$

so we see that W' and $W \otimes W$ are isomorphic as representations. Hence,

$$\chi_{\rho'} = \chi \cdot \chi = \chi^2.$$

\square

(c) Suppose that G is transitive on X and that X has at least two elements. We say that G is doubly transitive if, for all $x, y, x', y' \in X$ with $x \neq y$ and $x' \neq y'$, there exists $s \in G$ such that $x' = sx$ and $y' = sy$. Prove the equivalence of the following

- (i) G is doubly transitive.
- (ii) The action of G on $X \times X$ has two orbits, the diagonal and its complement.
- (iii) $(\chi^2|1) = 2$.
- (iv) The representation θ defined in (a) is irreducible.

Proof. [(i) \Leftrightarrow (ii)] Suppose G is doubly transitive and let D denote the diagonal in $X \times X$. If $x, y, x', y' \in X$ and $x \neq y$, $x' \neq y'$ then, by the definition of double transitivity, there exists $s \in G$ such that $s(x, y) = (sx, sy) = (x', y')$. If $(x, x), (y, y) \in D$, then, since G is transitive on X , there exists $s \in G$ such that $s(x, x) = (sx, sx) = (y, y)$, so the diagonal and its complement comprise the two orbits of $X \times X$. On the other hand, if the diagonal and its complement comprise the two orbits of $X \times X$, then it is immediately clear that G is doubly transitive, since for any $(x, y), (x', y') \in (X \times X) \setminus D$, there must exist an $s \in G$ such that $s(x, y) = (sx, sy) = (x', y')$.

[(ii) \Leftrightarrow (iii)] Note that the character of the permutation representation of $X \times X$ is χ^2 , as we saw in (b) above. By (a), $(\chi^2|1)$ is equal to the number

of orbits in $X \times X$. Hence, $(\chi^2|1) = 2$ if and only if there are precisely two orbits in $X \times X$.

[(iii) \Leftrightarrow (iv)] If ψ is the character of θ then, as we saw in (a), $1 + \psi = \chi$, $(1|1) = 1$ and $(\psi|1) = 0$. Now

$$\chi^2 = 1 + 2\psi + \psi^2,$$

so

$$(\chi^2|1) = 2 = (1|1) + 2(\psi|1) + (\psi^2|1) = 1 + (\psi^2|1) \Leftrightarrow (\psi^2|1) = 1.$$

This, in turn, is true iff

$$1 = (\psi^2|1) = \frac{1}{g} \sum_{s \in G} \psi^2(s) = (\psi|\psi)$$

since ψ is real valued, which is true iff θ is irreducible. Therefore, we see that θ is irreducible if and only if $(\chi^2|1) = 2$. \square

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