

## ALGEBRA HW 4

CLAY SHONKWILER

1

Prove that the number of non-isomorphic one-dimensional representations of a finite group  $G$  is  $|G|/|[G, G]|$ , where  $[G, G]$  denotes the commutator subgroup.

*Proof.* Suppose  $\phi : G \rightarrow \mathbb{C}^\times$  is a one-dimensional representation of  $G$ . Then we know that  $\phi$  factors through the quotient  $G/[G, G]$ , which is to say that, if  $\pi : G \rightarrow G/[G, G]$  is the natural map, then there exists a homomorphism  $\phi' : G/[G, G] \rightarrow \mathbb{C}^\times$  such that  $\phi = \phi' \circ \pi$ . In other words, each one-dimensional representation  $\phi$  of  $G$  corresponds to a one-dimensional representation  $\phi'$  of  $G/[G, G]$ . Furthermore, this correspondance is bijective. Surjectivity is clear; we see that this correspondance is injective since, if  $\rho, \chi$  are non-isomorphic one-dimensional representations of  $G$ , then there exists  $g \in G$  such that  $\rho(g) \not\cong \chi(g)$ ; then, if  $\pi(g) = \bar{g}$ ,  $\rho'(\bar{g}) \neq \chi'(\bar{g})$ .

Now, since  $G/[G, G]$  is abelian, it has  $|G/[G, G]| = |G|/|[G, G]|$  conjugacy classes, meaning it has  $|G|/|[G, G]|$  irreducible representations, each of dimension 1. Since each corresponds to a one-dimensional representation of  $G$ , we see that the number of one-dimensional representations of  $G$  is  $|G|/|[G, G]|$ .  $\square$

2

Let  $W_i$  be an irreducible representation of  $G$ ,  $V$  an arbitrary representation, and  $V_i$  the direct sum of all subrepresentations of  $V$  isomorphic to  $W_i$ . Let  $H_i$  be the space of linear maps  $f : W_i \rightarrow V$  such that  $\rho_V(g)f = f\rho(W_i)(g)$  for all  $g \in G$ .

(a) Show that the dimension of  $H_i$  is equal to the number of times that  $W_i$  occurs in  $V$ .

*Proof.* Suppose  $\phi \in H_i$ . Then  $\phi : W_i \rightarrow V$ . Now,  $V = \oplus U_j$ , so define  $p_j : V \rightarrow U_j$  to be the projection onto the  $j$ th coordinate.  $p_j$  is a homomorphism, so

$$\phi_j = p_j \circ \phi : W_i \rightarrow U_j$$

is a homomorphism of  $\mathbb{C}[G]$ -modules as well. Now, by Schur's Lemma, if  $W_i \simeq U_j$ , then  $\phi_j$  is an isomorphism given by multiplication by a scalar;

otherwise  $\phi_j$  is the trivial map. Hence,  $\text{Hom}(W_i, U_j)$  is isomorphic either to  $\mathbb{C}$  or to  $\{0\}$ . Now

$$H_i = \text{Hom}(W_i, V) = \text{Hom}(W_i, \oplus U_j) = \oplus \text{Hom}(W_i, U_j) = \oplus_{W_i \simeq U_j} \mathbb{C}.$$

From this, then, we conclude that the dimension of  $H_i$  is equal to the number of times that  $W_i$  occurs in  $V$ .  $\square$

(b) Let  $G$  act on  $H_i \otimes W_i$  through the tensor product of the trivial representation on  $H_i$  and the given representation on  $W_i$ . Show that the map:

$$F : H_i \otimes W_i \rightarrow V_i$$

defined by

$$F\left(\sum T_\alpha \otimes w_\alpha\right) = \sum T_\alpha(w_\alpha)$$

is an isomorphism of  $H_i \otimes W_i$  onto  $V_i$ .

*Proof.* We must show that  $F$  commutes with the action of  $G$  on  $H_i \otimes W_i$  and on  $V_i$ . That is, we must show the following diagram commutes:

$$\begin{array}{ccc} H_i \otimes W_i & \xrightarrow{F} & V_i \\ \rho' = 1_{H_i} \otimes \rho_{W_i} \downarrow & & \downarrow \rho|_{V_i} \\ H_i \otimes W_i & \xrightarrow{F} & V_i. \end{array}$$

If  $f \in H_i$ ,  $w \in W_i$ ,  $g \in G$  then

$$\begin{aligned} (F \circ (1_H \otimes \rho_{W_i})(g))(f \otimes w) &= F(f \otimes \rho_{W_i}(g)(w)) \\ &= f \rho_{W_i}(g)(w) \\ &= \rho_V(g)f(w) \\ &= \rho|_{V_i}(g)f(w) \\ &= (\rho|_{V_i} \circ F)(f \otimes w), \end{aligned}$$

by the properties of  $f$ , so the diagram above does indeed commute.

Now, suppose  $\sum v_\alpha \in V_i$ . Then each  $v_\alpha$  lies in an isomorphic copy of  $W_i$  in  $V$ , there exists an isomorphism  $T_\alpha$  and an element  $w_\alpha \in W_i$  such that  $T_\alpha(w_\alpha) = v_\alpha$ . Hence,

$$F\left(\sum T_\alpha \otimes w_\alpha\right) = \sum T_\alpha(w_\alpha) = \sum v_\alpha.$$

Therefore,  $F$  is surjective. Now, to see that  $F$  is injective, and therefore an isomorphism, we need only count dimensions.

$$\dim(H_i \otimes W_i) = \dim(H_i) \cdot \dim(W_i).$$

On the other hand, as we showed in part (a),  $V_i$  is the direct sum of  $\dim(H_i)$  copies of  $W_i$ , each of dimension  $\dim(W_i)$ , so

$$\dim(V_i) = \dim(H_i) \cdot \dim(W_i).$$

Since  $F$  is a surjective linear map between vector spaces of the same dimension, it must be injective as well.  $\square$

## 3

Let  $\chi$  be the character of 2-dimensional representation of  $G$ , and let  $x$  be an element of order 2 in  $G$ . Prove that  $\chi(x) = 2, 0, -2$ . Generalize this to  $n$ -dimensional representations.

*Proof.* Let  $\rho : B \rightarrow V$  be the 2-dimensional representation. Note that we can think of  $V$  as being simply  $\mathbb{C}^2$ . Now, suppose

$$\rho(x) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Then  $\rho^2(x) = \rho(x^2) = \rho(1) = \text{Id}_2$ . Also, note that  $\chi(x) = \text{Tr}(\rho(x)) = a + d$ . Squaring the matrix for  $\rho(x)$ , then, we see that the following system of equations must hold:

$$\begin{aligned} a^2 + bc &= 1 \\ ab + bd &= 0 \\ ac + cd &= 0 \\ bc + d^2 &= 1. \end{aligned}$$

We note that the bottom equation implies  $bc = 1 - d^2$ , which, replacing for  $bc$  in the top equation yields

$$1 = a^2 + 1 - d^2 \Rightarrow 0 = a^2 - d^2 = (a + d)(a - d).$$

If  $a + d = 0$ , then  $\chi(x) = \text{Tr}(\rho(x)) = a + d = 0$ . On the other hand, if  $a - d = 0$ , then  $a = d$ . Substituting into the second equation in the above system, we see that

$$0 = ab + ba = 2ab.$$

If  $a = 0$ , then  $d = a = 0$ , so  $\chi(x) = a + d = 0$ . If  $b = 0$ , then, from the first equation, we see that

$$a^2 = 1 \Rightarrow a = \pm 1.$$

Hence  $d = a = \pm 1$ , so  $\chi(x) = a + d = \pm 2$ . Since we've covered all the possibilities, we see that, indeed,  $\chi(x) = 2, 0, -2$ .  $\square$

## 4

Let  $\chi$  be an irreducible character of  $G$ . Prove that for every element  $g$  in  $Z(G)$ , the center of  $G$ , we have  $\chi(g) = \epsilon\chi(1)$ , where  $\epsilon$  is a root of unity.

*Proof.* Let  $g \in Z(G)$ . Let  $\rho : G \rightarrow GL(V)$  be the irreducible representation with character  $\chi$ . Then, for any  $x \in G$ ,

$$\rho(g) \cdot \rho(x) = \rho(gx) = \rho(xg) = \rho(x) \cdot \rho(g).$$

Since  $\rho$  is irreducible and  $\rho(g)$  is certainly a linear map from  $V$  to itself, we can use Schur's Lemma to conclude that  $\rho(g)$  is simply a scalar multiple of the identity;

$$\rho(g) = \lambda \text{Id}.$$

Hence

$$\chi(g) = \text{Tr}(\rho(g)) = \text{Tr}(\lambda \text{Id}) = \lambda \text{Tr}(\text{Id}) = \lambda \chi(1).$$

Now, we note that, since  $G$  is finite, there exists  $n \in \mathbb{N}$  such that  $g^n = 1$ . Hence

$$\text{Id} = \rho(g^n) = (\rho(g))^n = (\lambda \text{Id})^n = \lambda^n \text{Id},$$

so  $\lambda^n = 1$ . Therefore,  $\lambda$  is a root of unity.  $\square$

## 5

Let  $\chi$  be the character of some representation  $\rho$  of  $G$ . Prove that

(a) If  $\chi(g) = \chi(1)$  then  $g \in \text{Ker}(\rho)$ .

*Proof.* Let  $g \in G$ . We choose an orthonormal basis for  $V$  such that  $\rho(g)$  is unitary. Then

$$\text{Tr}(\rho(g)) = \chi(g) = \chi(1) = \text{Tr}(\text{Id}) = n,$$

where  $n$  is the dimension of  $V$ . However, the only unitary matrix with trace  $n$  is the identity itself, so we see that  $\rho(g) = \text{Id}$ . However, this is precisely the condition for  $g$  to be in the kernel of  $\rho$ .  $\square$

(b) If  $|\chi(g)| = \chi(1)$  and  $\rho$  is faithful, then  $g \in Z(G)$  (the center of  $G$ ).

*Proof.* Let  $g \in G$  such that  $|\chi(g)| = \chi(1)$ . Then there exists an orthonormal basis with respect to which  $\rho(g)$  is diagonal. Furthermore, since the minimal polynomial of  $\rho(g)$  must divide the polynomial  $x^{|g|} - 1$  and the diagonal elements must be roots of the minimal polynomial (eigenvalues), the elements on the diagonal must be some set of roots of unity,  $\{\zeta_1, \dots, \zeta_n\}$ , where  $n$  is the dimension of the representation.

Now,

$$n = \chi(1) = |\chi(g)| = |\text{Tr}(\rho(g))| = |\zeta_1 + \dots + \zeta_n|.$$

The only way this can be true is if  $\zeta_1 = \zeta_2 = \dots = \zeta_n$ . Hence, still with respect to the chosen basis,

$$\rho(g) = \zeta \text{Id}$$

for some root of unity  $\zeta$ . Now, let  $x \in G$ . Then

$$\rho(xgx^{-1}) = \rho(x)\rho(g)(\rho(x))^{-1} = \rho(x)\zeta \text{Id}(\rho(x))^{-1} = \zeta \rho(x)(\rho(x))^{-1} = \zeta \text{Id} = \rho(g).$$

Now, since  $\rho$  is faithful, this implies that  $g = xgx^{-1}$ . Since our choice of  $x$  was arbitrary, we see that this holds for all  $x \in G$ , meaning  $g \in Z(G)$ .  $\square$

DRL 3E3A, UNIVERSITY OF PENNSYLVANIA

*E-mail address:* shonkwil@math.upenn.edu