MATH 560 FINAL EXAM

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1

Consider the surface

$$\mathbf{X}(u,v) = \left(u + uv^2 - \frac{u^3}{3}, v + u^2v - \frac{v^3}{3}, u^2 - v^2\right).$$

(a): Calculate the first and second fundamental forms of this surface and show that its mean curvature is zero.

Answer: We know that $E = \langle \mathbf{X}_u, \mathbf{X}_u \rangle$, $F = \langle \mathbf{X}_u, \mathbf{X}_v \rangle$ and $G = \langle \mathbf{X}_v, \mathbf{X}_v \rangle$, so we need to calculate $\mathbf{X}_u$ and $\mathbf{X}_v$. Now,

$$\mathbf{X}_u = (1 - u^2 + v^2, 2uv, 2u)$$

and

$$\mathbf{X}_v = (2uv, 1 - v^2 + u^2, -2v).$$

Hence,

$$\begin{aligned} E = \langle \mathbf{X}_u, \mathbf{X}_u \rangle &= (1 - u^2 + v^2)^2 + 4u^2v^2 + 4u^2 \\ &= 1 + 2u^2 + 2v^2 + u^4 + v^4 2u^2v^2 \\ &= (1 + u^2 + v^2)^2 \end{aligned}$$

$$\begin{aligned} F = \langle \mathbf{X}_u, \mathbf{X}_v \rangle &= 2uv(1 - u^2 + v^2) + 2uv(1 - v^2 + u^2) - 4uv \\ &= 2uv - 2u^3v + 2uv^3 + 2uv - 2uv^3 + 2u^3v - 4uv \\ &= 0 \end{aligned}$$

and

$$\begin{aligned} G = \langle \mathbf{X}_v, \mathbf{X}_v \rangle &= 4u^2v^2 + (1 - v^2 + u^2)^2 + 4v^2 \\ &= 1 + 2u^2 + 2v^2 + u^4 + v^4 2u^2v^2 \\ &= (1 + u^2 + v^2)^2 \end{aligned}.$$

Now,

$$\mathbf{N} = \frac{\mathbf{X}_u \times \mathbf{X}_v}{||\mathbf{X}_u \times \mathbf{X}_v||} = \frac{\mathbf{X}_u \times \mathbf{X}_v}{\sqrt{EG - F^2}}$$

$$= \frac{(-2uv^2 - 2u - 2u^3, 2u^2v + 2v + 2v^3, 1 - 2u^2v^2 - u^4 - v^4)}{\sqrt{(1 + u^2 + v^2)^4}}$$

Hence,

$$\begin{aligned}
L &= \langle \mathbf{N}, \mathbf{X}_{uu} \rangle \\
&= \tfrac{1}{(1+u^2+v^2)^2}\langle(-2uv^2-2u-2u^3, 2u^2v+2v+2v^3, 1-2u^2v^2-u^4-v^4), \\
&\quad (-2u, 2v, 2)\rangle \\
&= \tfrac{1}{(1+u^2+v^2)^2} 2(1+2u^2+2v^2+u^4+v^4+2u^2v^2) \\
&= 2\tfrac{(1+u^2+v^2)^2}{(1+u^2+v^2)^2} \\
&= 2.
\end{aligned}$$

Now,

$$\begin{aligned}
M &= \langle \mathbf{N}, \mathbf{X}_{uv} \rangle \\
&= \tfrac{1}{(1+u^2+v^2)^2}\langle(-2uv^2-2u-2u^3, 2u^2v+2v+2v^3, \\
&\quad 1-2u^2v^2-u^4-v^4), (2v, 2u, 0)\rangle \\
&= \tfrac{1}{(1+u^2+v^2)^2} 2(-2uv^3-2uv+2u^3v-2u^3v+2uv+2uv^3) \\
&= \tfrac{1}{(1+u^2+v^2)^2}(0) \\
&= 0.
\end{aligned}$$

and

$$\begin{aligned}
N &= \langle \mathbf{N}, \mathbf{X}_{vv} \rangle \\
&= \tfrac{1}{(1+u^2+v^2)^2}\langle(-2uv^2-2u-2u^3, 2u^2v+2v+2v^3, \\
&\quad 1-2u^2v^2-u^4-v^4), (2u, -2v, -2)\rangle \\
&= -\tfrac{1}{(1+u^2+v^2)^2}\langle(-2uv^2-2u-2u^3, 2u^2v+2v+2v^3, \\
&\quad 1-2u^2v^2-u^4-v^4), (2u, -2v, -2)\rangle \\
&= -L \\
&= -2.
\end{aligned}$$

That is to say,

$$L = 2, \quad N = -2, \quad M = 0.$$

To compute the mean curvature, we calculate the principal curvatures:

$$k_1 = -a_{11} = \frac{-MF+LG}{EG-F^2}, \quad k_2 = -a_{22} = \frac{-MF+NE}{EG-F^2}.$$

Since $M = F = 0$,

$$k_1 = \frac{LG}{EG} = \frac{L}{E}, \quad k_2 = \frac{NE}{EG} = \frac{N}{G}.$$

Hence,

$$k_1 = \frac{2}{(1+u^2+v^2)^2}, \quad k_2 = \frac{-2}{(1+u^2+v^2)^2}.$$

Therefore,

$$H = k_1 + k_2 = \frac{2}{(1+u^2+v^2)^2} - \frac{2}{(1+u^2+v^2)^2} = 0.$$

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(b): Find the parametric equations of the conjugate minimal surface of $\mathbf{X}(u,v)$.

Answer: Denote Enneper's surface as parametrized in (a) by

$$\mathbf{X}(u,v) = (\mathbf{x}(u,v), \mathbf{y}(u,v), \mathbf{z}(u,v)).$$

Then the conjugate minimal surface to Enneper's surface is

$$\boldsymbol{\Xi}(u,v) = (\xi(u,v), \eta(u,v), \zeta(u,v))$$

where

$$\begin{aligned}
\xi_v &= \frac{1}{w}\left(G\mathbf{x}_u - F\mathbf{x}_v\right)\\
\xi_u &= \frac{1}{w}\left(-F\mathbf{x}_u - E\mathbf{x}_v\right)\\
\eta_v &= \frac{1}{w}\left(G\mathbf{y}_u - F\mathbf{y}_v\right)\\
\eta_u &= \frac{1}{w}\left(-F\mathbf{y}_u - E\mathbf{y}_v\right)\\
\zeta_v &= \frac{1}{w}\left(G\mathbf{z}_u - F\mathbf{z}_v\right)\\
\zeta_u &= \frac{1}{w}\left(-F\mathbf{z}_u - E\mathbf{z}_v\right)
\end{aligned}$$

where E, F, G are as in (a) and $w = \sqrt{EG - F^2}$. Now, since we saw in (a) that $E = G = (1 + u^2 + v^2)^2$ and $F = 0$, $w = E = G$ and the above reduce to

$$\begin{aligned}
\xi_v &= \mathbf{x}_u = 1 + v^2 - u^2\\
\xi_u &= -\mathbf{x}_v = -2uv\\
\eta_v &= \mathbf{y}_u = 2uv\\
\eta_u &= -\mathbf{y}_v = -1 - u^2 + v^2\\
\zeta_v &= \mathbf{z}_u = 2u\\
\zeta_u &= -\mathbf{z}_v = 2v,
\end{aligned}$$

so it's easy to see that

$$\boldsymbol{\Xi}(u,v) = \left(v - u^2 v + \frac{v^3}{3}, -u + uv^2 - \frac{u^3}{3}, 2uv\right).$$

A simple computation (which I won't reproduce here) confirms that this surface has the same first fundamental form as Enneper's surface and mean curvature zero, so it really is a minimal surface.

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(c): Prove Enneper's surface $\{\mathbf{X}(u,v); -\infty < u < \infty, -\infty < v < \infty\}$ is complete.

Answer: I honestly have no idea how to prove this, despite having spent rather a long time trying to figure it out.

2

(a): Prove that developable surfaces $\{\mathbf{X}(u,v) = \mathbf{x}(u) + v\mathbf{V}(u)\}$ are characterized by the condition (added to the regularity assumptions) $(\dot{\mathbf{x}}(u), \mathbf{V}(u), \dot{\mathbf{V}}(u)) = 0$, identically.

Proof. In order to show this, we show that each characterization of devolopability implies the other.

Suppose S is a developable surface with parametrization $\mathbf{X}(u,v) = \mathbf{x}(u) + v\mathbf{V}(u)$. Let $\mathbf{X}(u_0, v_0)$ be a point of S. Then, since S is developable, the tangent plane $T_{\mathbf{X}(u_0,v_0)}S$ is tangent to D all along the line $v \mapsto \mathbf{X}(u_0, v_0) + v\mathbf{V}(u_0)$. Hence, the normal $\mathbf{N}$ to S is constant along this line; since this is true for all choices of u_0 and v_0, we see that $\mathbf{N}_v \equiv 0$ on all of S. In particular, this implies that

$$M = \langle \mathbf{N}, \mathbf{X}_{uv} \rangle = -\langle \mathbf{N}_v, \mathbf{X}_u \rangle = 0.$$

In order to use this information, we need to compute $\mathbf{N}$ and $\mathbf{X}_{uv}$ explicitly. To that end, note that $\mathbf{X}_u = \dot{\mathbf{x}}(u) + v\dot{\mathbf{V}}(u)$ and $\mathbf{X}_v = \mathbf{V}(u)$. Hence,

$$\begin{aligned} \mathbf{X}_u \times \mathbf{X}_v &= (\dot{\mathbf{x}}(u) + v\dot{\mathbf{V}}(u)) \times \mathbf{V}(u) \\ &= \dot{\mathbf{x}}(u) \times \mathbf{V}(u) + v\dot{\mathbf{V}}(u) \times \mathbf{V}(u), \end{aligned}$$

so

$$\mathbf{N} = \frac{\dot{\mathbf{x}}(u) \times \mathbf{V}(u) + v\dot{\mathbf{V}}(u) \times \mathbf{V}(u)}{\|\dot{\mathbf{x}}(u) \times \mathbf{V}(u) + v\dot{\mathbf{V}}(u) \times \mathbf{V}(u)\|}, \tag{1}$$

which is well-defined since $\mathbf{X}_u \times \mathbf{X}_v \neq 0$ (by the regularity condition). Note that this calculation depended only on the fact that S is a ruled surface.

Now,

$$\mathbf{X}_{uv} = \dot{\mathbf{V}}(u),$$

so

$$\begin{aligned} 0 &= M\langle \mathbf{N}, \mathbf{X}_{uv} \rangle \\ &= \frac{1}{\|\dot{\mathbf{x}}(u) \times \mathbf{V}(u) + v\dot{\mathbf{V}}(u) \times \mathbf{V}(u)\|} \langle \dot{\mathbf{V}}(u), \dot{\mathbf{x}}(u) \times \mathbf{V}(u) + v\dot{\mathbf{V}}(u) \times \mathbf{V}(u) \\ &= \frac{1}{\|\dot{\mathbf{x}}(u) \times \mathbf{V}(u) + v\dot{\mathbf{V}}(u) \times \mathbf{V}(u)\|} (\dot{\mathbf{x}}(u), \mathbf{V}(u), \dot{\mathbf{V}}(u)). \end{aligned}$$

Therefore, it must be the case that $(\dot{\mathbf{x}}(u), \mathbf{V}(u), \dot{\mathbf{V}}(u)) = 0$.

On the other hand, suppose S is a ruled surface parametrized by $\mathbf{X}(u,v) = \mathbf{x}(u) + v\mathbf{V}(u)$, satisfies the given regularity conditions and is such that $(\dot{\mathbf{x}}(u), \mathbf{V}(u), \dot{\mathbf{V}}(u)) = 0$. Then, as in (1),

$$\mathbf{N} = \frac{\dot{\mathbf{x}}(u) \times \mathbf{V}(u) + v\dot{\mathbf{V}}(u) \times \mathbf{V}(u)}{\|\dot{\mathbf{x}}(u) \times \mathbf{V}(u) + v\dot{\mathbf{V}}(u) \times \mathbf{V}(u)\|}$$

so

$$\langle \mathbf{N}_v, \mathbf{X}_u \rangle = -\langle \mathbf{N}, \mathbf{X}_{uv} \rangle = -\frac{1}{\|\dot{\mathbf{x}}(u) \times \mathbf{V}(u) + v\dot{\mathbf{V}}(u) \times \mathbf{V}(u)\|} (\dot{\mathbf{x}}(u), \mathbf{V}(u), \dot{\mathbf{V}}(u)) = 0$$

and

$$\langle \mathbf{N}_v, \mathbf{X}_v \rangle = -\langle \mathbf{N}, \mathbf{X}_{vv} \rangle = -\langle \mathbf{N}, 0 \rangle = 0.$$

Hence, since $\mathbf{X}_u$ and $\mathbf{X}_v$ are linearly independent, we conclude that $\mathbf{N}_v \equiv 0$ identically, which means that $\mathbf{N}$ is constant along the lines $v \mapsto \mathbf{X}(u_0, v_0) + v\mathbf{V}(u_0)$, which in turn implies that $T_{\mathbf{X}(u_0,v_0)}S$ is tangent to S all along this line for any u_0, v_0. Therefore, we see that S is a developable surface.

Thus, having shown implications both ways, we conclude that the two conditions for being developable given are equivalent. □

(b): We now consider ruled surfaces that are *strictly not developable*, meaning that, for each u, $(\dot{\mathbf{x}}(u), \mathbf{V}(u), \dot{\mathbf{V}}(u)) \neq 0$. Show that a strictly non-developable ruled surface S has no local singularities and that its Gaussian curvature is everywhere strictly negative.

Proof. Note that our computations for $\mathbf{X}_u, \mathbf{X}_v$ and M in part (a) are valid for any ruled surface, so we can import them directly to this problem. Now, $\mathbf{X}_{vv} = (\mathbf{X}_v)_v = (\mathbf{V}(u))_v = 0$, so $N = 0$. Hence,

$$K = \frac{LN - M^2}{EG - F^2} = \frac{-M^2}{EG - F^2},$$

so we don't even need to bother computing L. Using the value for M computed in (a), we have that

$$K = \frac{-M^2}{EG - F^2} = -\frac{1}{\|\dot{\mathbf{x}}(u) \times \mathbf{V}(u) + v\dot{\mathbf{V}}(u) \times \mathbf{V}(u)\|} \frac{(\dot{\mathbf{x}}(u), \mathbf{V}(u), \dot{\mathbf{V}}(u))^2}{EG - F^2}.$$

Since $\|\mathbf{X}_u \times \mathbf{X}_v\| = \|\dot{\mathbf{x}}(u) \times \mathbf{V}(u) + v\dot{\mathbf{V}}(u) \times \mathbf{V}(u)\|$ and $(\dot{\mathbf{x}}(u), \mathbf{V}(u), \dot{\mathbf{V}}(u))^2$ are both strictly positive, if we can show that $EG - F^2 > 0$, then that will suffice to show that $K < 0$. However,

$$EG - F^2 = \|\mathbf{X}_u\|^2 \|\mathbf{X}_v\|^2 - \langle \mathbf{X}_u, \mathbf{X}_v \rangle^2 = \|\mathbf{X}_u \times \mathbf{X}_v\|^2 > 0.$$

Therefore, we conclude that, indeed, $K < 0$. □

(c): We shall assume now that the parameter u of the generating curve $\Gamma_0 = \{\mathbf{x}(u)\}$ of S describes its oriented arc length ($\dot{\mathbf{x}} \cdot \dot{\mathbf{x}} = 1$) and that the direction $\mathbf{V}(u)$ of the rulings is everywhere normal to Γ_0, *i.e.* $\mathbf{V}(u) = \mathbf{n}(u)\cos\theta(u) + \mathbf{b}\sin\theta(u)$, where $\mathbf{n}$ denotes the principal normal and $\mathbf{b}$ the binormal of Γ_0. Prove that S is strictly non-developable if and only if

$$\frac{d\theta(u)}{du} + \tau(u) \neq 0,$$

where $\tau(u)$ denotes the torsion of Γ_0.

Proof. Suppose S is strictly non-developable. Then $(\dot{\mathbf{x}}(u), \mathbf{V}(u), \dot{\mathbf{V}}(u)) \neq 0$. Now,

$$\mathbf{V}(u) = \mathbf{n}(u)\cos\theta(u) + \mathbf{b}(u)\sin\theta(u),$$

so

$$\begin{aligned}\dot{\mathbf{V}}(u) &= \dot{\mathbf{n}}(u)\cos\theta(u) - \mathbf{n}(u)\dot{\theta}(u)\sin\theta(u) + \dot{\mathbf{b}}(u)\sin\theta(u) + \mathbf{b}(u)\dot{\theta}(u)\cos\theta(u)\\ &= (-\kappa\dot{\mathbf{x}}(u) + \tau\mathbf{b}(u))\cos\theta(u) - \mathbf{n}\dot{\theta}(u)\sin\theta(u) - \tau\mathbf{n}(u)\sin\theta(u) + \mathbf{b}(u)\dot{\theta}(u)\cos\theta(u)\\ &= -\kappa\cos\theta(u)\dot{\mathbf{x}}(u) - (\tau + \dot{\theta}(u))\sin\theta(u)\mathbf{n}(u) + (\dot{\theta}(u) + \tau)\cos\theta(u)\mathbf{b}(u),\end{aligned}$$

by the Frenet equations. Therefore,

$$\mathbf{V}(u) \times \dot{\mathbf{V}}(u) = (\dot{\theta}(u) + \tau)\dot{\mathbf{x}}(u) - \kappa\sin\theta(u)\cos\theta(u)\mathbf{n}(u) + \kappa\cos^2\theta(u)\mathbf{b}(u),$$

so

$$\begin{aligned}0 &\neq (\dot{\mathbf{x}}(u), \mathbf{V}(u), \dot{\mathbf{V}}(u))\\ &= \dot{\mathbf{x}}(u)\cdot(\mathbf{V}(u)\times\dot{\mathbf{V}}(u))\\ &= (\dot{\theta}(u) + \tau)\|\dot{\mathbf{x}}(u)\|^2\\ &= \frac{d\theta(u)}{du} + \tau.\end{aligned}$$

Note that all of the equalities in the above expression hold regardless of any assumptions we made about S, so we see that, if S is a surface satisfying the given hypothesis and such that $\frac{d\theta(u)}{du} + \tau(u) \neq 0$, then S is strictly non-developable. □

(d): With the same assumptions as in (c), prove that, if $\theta(u) = 0$ (identically) and $\tau(u) \neq 0$, then the mean curvature $H(u, v)$ of S equals 0 all along Γ_0.

Proof. Suppose this is the case. Then $\mathbf{V}(u) = \mathbf{n}(u)$, so $\mathbf{X}(u, v) = \mathbf{x}(u) + v\mathbf{n}(u)$. Hence, along the curve Γ_0 (*i.e.* where $v = 0$),

$$\mathbf{X}_u = \dot{\mathbf{x}}(u) + v\dot{\mathbf{n}}(u) = \dot{\mathbf{x}}(u)$$

and

$$\mathbf{X}_v = \mathbf{n}(u).$$

Now, note that $F = \langle\mathbf{X}_u, \mathbf{X}_v\rangle = \dot{\mathbf{x}}(u), \mathbf{n}(u)\rangle = 0$. Moreover,

$$\begin{aligned}\mathbf{X}_{uu} &= \ddot{\mathbf{x}}(u)\\ \mathbf{X}_{uv} &= \dot{\mathbf{n}}(u)\\ \mathbf{X}_{vv} &= 0\end{aligned}$$

and

$$\mathbf{N} = \frac{\mathbf{X}_u \times \mathbf{X}_v}{\|\mathbf{X}_u \times \mathbf{X}_v\|} = \frac{\dot{\mathbf{x}}(u) \times \mathbf{n}(u)}{\|\dot{\mathbf{x}}(u) \times \mathbf{n}(u)\|} = \mathbf{b}(u).$$

Therefore, $N = 0$ and

$$L = \langle\mathbf{N}, \mathbf{X}_{uu}\rangle = \langle\mathbf{b}(u), \dot{\mathbf{x}}(u)\rangle = 0,$$

so

$$H = \frac{1}{2}\frac{LG - 2MF + NE}{EG - F^2} = 0$$

since $L = F = N = 0$. $\square$

(e): (Challenge) Calculate $H(u,v)$ under the same assumptions of questions (b), (c), (d). In particular, show that, if the curvature $\kappa(u)$ of Γ_0 is positive, then there is a positive valued function $f(u)$ such that $H(u,v) = 0$, if and only if either $v = 0$ or $v = f(u)$.

Partial Answer: Suppose S is as in (b), (c) and (d); that is, S is a strictly non-developable ruled surface with $\|\dot{\mathbf{x}}\| = 1$ and $\mathbf{V}(u) = \mathbf{n}(u)$. Note that, since S is strictly non-developable, (c) implies that $\tau(u) \neq 0$ (since $\theta(u) = 0$). Now, under these assumptions, $\mathbf{X}(u,v) = \mathbf{x}(u) + v\mathbf{V}(u) = \mathbf{x}(u) + v\mathbf{n}(u)$. Hence,

$$\mathbf{X}_u = \dot{\mathbf{x}}(u) + v\dot{\mathbf{n}}(u) = \dot{\mathbf{x}}(u) + v(-\kappa(u)\dot{\mathbf{x}}(u) + \tau(u)\mathbf{b}(u)) = (1 - v\kappa(u))\dot{\mathbf{x}}(u) + v\tau(u)\mathbf{b}(u)$$
$$\mathbf{X}_v = \mathbf{n}(u).$$

Hence,

$$\mathbf{X}_u \times \mathbf{X}_v = (1 - v\kappa(u))\mathbf{b}(u) - v\tau(u)\dot{\mathbf{x}}(u),$$

so

$$\mathbf{N} = \frac{\mathbf{X}_u \times \mathbf{X}_v}{\|\mathbf{X}_u \times \mathbf{X}_v\|} = \frac{1}{v^2\tau^2 + (1 - v\kappa)^2}\left[-v\tau(u)\dot{\mathbf{x}}(u) + (1 - v\kappa(u))\mathbf{b}(u)\right].$$

Also,

$$\begin{aligned}\mathbf{X}_{uu} &= \ddot{\mathbf{x}}(u) + v(-\dot{\kappa}(u)\dot{\mathbf{x}}(u) - \kappa(u)\ddot{\mathbf{x}}(u) + \dot{\tau}(u)\mathbf{b}(u) + \tau(u)\dot{\mathbf{b}}(u) \\ &= -v\dot{\kappa}(u)\dot{\mathbf{x}}(u) + (\kappa(u) - v\kappa^2(u) - \tau^2(u))\mathbf{n}(u) + v\dot{\tau}(u)\mathbf{b}(u),\end{aligned}$$

while

$$\mathbf{X}_{uv} = -\kappa(u)\dot{\mathbf{x}}(u) + \tau(u)\mathbf{b}(u)$$

and

$$\mathbf{X}_{vv} = 0.$$

Hence,

$$\begin{aligned}L &= \langle \mathbf{N}, \mathbf{X}_{uu}\rangle \\ &= \frac{\langle -v\tau(u)\dot{\mathbf{x}}(u) + (1 - v\kappa(u))\mathbf{b}(u), -v\dot{\kappa}(u)\dot{\mathbf{x}}(u) + (\kappa(u) - v\kappa^2(u) - \tau^2(u))\mathbf{n}(u) + v\dot{\tau}(u)\mathbf{b}(u)\rangle}{v^2\tau^2 + (1 - v\kappa)^2} \\ (2) \quad &= \frac{v^2\tau(u)\dot{\kappa}(u) + v\dot{\tau}(u) - v^2\dot{\tau}(u)\kappa(u)}{v^2\tau^2 + (1 - v\kappa)^2}.\end{aligned}$$

Also, note that $N = 0$, $F = \langle \mathbf{X}_u, \mathbf{X}_v\rangle = 0$, $G = 1$ and

$$\begin{aligned}E &= \langle \mathbf{X}_u, \mathbf{X}_u\rangle \\ &= \langle (1 - v\kappa(u))\dot{\mathbf{x}}(u) + v\tau(u)\mathbf{b}(u), (1 - v\kappa(u))\dot{\mathbf{x}}(u) + v\tau(u)\mathbf{b}(u)\rangle \\ &= (1 - v\kappa(u))^2 + v^2\tau^2(u).\end{aligned}$$

Therefore,

$$H = \frac{1}{2}\frac{LG - 2MF + NE}{EG - F^2} = \frac{LG}{2EG} = \frac{L}{2E},$$

which is zero if and only if either $v = 0$ (which was part (d) above) or $L = 0$, which, given (2), means that

$$0 = v^2\tau(u)\dot{\kappa}(u) + v\dot{\tau}(u) - v^2\dot{\tau}(u)\kappa(u).$$

In turn, this equality holds if and only if

$$v = \frac{\dot{\tau}(u)}{\dot{\tau}(u)\kappa(u) - \tau(u)\dot{\kappa}(u)}.$$

I believe the righthand side of this equality is a positive function, but I can't prove it. Assuming it is, this would demonstrate that $H = 0$ if and only if either $v = 0$ or $V = f(u)$ for some positive function $f(u)$.

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