MATH 570/PHIL 006/506 FINAL

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1

Show that $\omega \longrightarrow (\omega)^2_2$.

Proof. Let $f : [\omega]^2 \to \{0, 1\}$. Define

$$A_0 = \{a \in \omega | a > 0 \text{ and } f(\{0, a\}) = 0\}$$

and

$$A_1 = \{a \in \omega | a > 0 \text{ and } f(\{0, a\}) = 1\}.$$

Then A_0 and A_1 partition $\omega - \{0\}$; since $\omega - \{0\}$ is infinite, either A_0 or A_1 (or possibly both) must be infinite. Let $A' = A_0$ if A_0 is infinite and let $A' = A_1$ otherwise. Let i_1 be the least element of A'. Define

$$A'_0 = \{a \in A' | a > i_1 \text{ and } f(\{i_1, a\}) = 0\}$$

and

$$A'_1 = \{a \in A' | a > i_1 \text{ and } f(\{i_1, a\}) = 1\}.$$

Then A'_0 and A'_1 partition $A' - \{i_1\}$; since $A' - \{i_1\}$ is infinite, either A'_0 or A'_1 (or possibly both) must be infinite. Let $A'' = A'_0$ if A'_0 is infinite and let $A'' = A'_1$ otherwise. Let i_2 be the smallest element of A'' and iterate this process.

Thus, we produce an infinite sequence

$$\omega \supset A' \supset A'' \supset \cdots$$

where $A^{(k)}$ is infinite for each $k \in \omega$ and has least element i_k. Let

$$M = \{0, i_1, i_2, \dots\}$$

and let $i_0 = 0$. Now, since, for each $n \in \omega$, $i_n \in A^{(m)}$ for all $m \leq n$, we see that for each choice of j, $f(\{i_j, i_p\}) = f(\{i_j, i_q\})$ for all $p, q > j$. Hence, we can define a partition on the natural numbers by coloring each such number j red if $f(\{i_j, i_n\}) = 0$ for all $n > j$ and blue if $f(\{i_j, i_n\}) = 1$ for all $n > j$. Since ω is infinite, any partition of the elements of ω must consist of at least one infinite set, so there must be infinitely many red numbers or infinitely many blue numbers according to our coloring scheme. Suppose, without loss of generality, that there are infinitely many red numbers. Then define

$$M' = \{i_n \in M | n \text{ is red}\}.$$

Then f is constant on $[M']^2 \subset [\omega]^2$. Since our choice of f was arbitrary, we see that, indeed,

$$\omega \longrightarrow (\omega)^2_2.$$

□

2

Show that for all $k \in \omega$, there is an $r \in \omega$ such that $r \longrightarrow (k)^2_2$.

Proof. Let $k \in \omega$ and let $r = 2^{2k-1}$. Let $f : [r]^2 \to \{0, 1\}$. Now, let

$$A_0 = \{a \in r | a > 0 \text{ and } f(\{0, a\}) = 0\}$$

and

$$A_1 = \{a \in r | a > 0 \text{ and } f(\{0, a\}) = 1\}.$$

Let A' be the bigger of A_0 and A_1. Then, since $r - \{0\}$ has cardinality $2^{2k-1} - 1$, A' must have cardinality at least 2^{2k-2}. Now, let i_1 be the smallest element of A' and let

$$A'_0 = \{a \in A' | a > i_1 \text{ and } f(\{i_1, a\}) = 0\}$$

and

$$A'_1 = \{a \in A' | a > i_1 \text{ and } f(\{i_1, a\}) = 1\}.$$

Let A'' be the bigger of A'_0 and A'_1. Then, since $A' - \{i_1\}$ has cardinality at least $2^{2k-2} - 1$, A'' mujst have cardinality at least 2^{2k-3}. Let i_2 be the smallest element of A'' and iterate this process.

At each stage, we produce an $A^{(m)}$ with cardinality at least 2^{2k-1-m} and least element i_m. Eventually, we terminate this process at $A^{(2k-1)}$, which has cardinality at least $2^{2k-1-(2k-1)} = 2^0 = 1$ and least element i_{2k-1}. Now, consider the set

$$M = \{0, i_1, i_2, \dots, i_{2k-1}\},$$

which has cardinality $2k$. Let $i_0 = 0$. Now, since, for each $n = 1, \dots, 2k-1$, $i_n \in A^{(m)}$ for all $m \leq n$, we see that for each choice of j, $f(\{i_j, i_p\}) = f(\{i_j, i_q\})$ for all $p, q > j$. Hence, we can define a partition on the natural numbers less than $2k$ (including 0) by coloring each such number j red if $f(\{i_j, i_n\}) = 0$ for all $n > j$ and blue if $f(\{i_j, i_n\}) = 1$ for all $n > j$. Since there are $2k$ such numbers and only two choices of color, we see that there must be at least k of these numbers with the same coloration. Suppose, without loss of generality, that there are at least k of these numbers colored red. Then define

$$M' = \{i_m \in M | m \text{ is red}\}.$$

Then f is constant on $[M']^2 \subset [r]^2$. Since our choice of f was arbitrary, we see that, indeed,

$$r \longrightarrow (k)^2_2.$$

□

3

A binary tree T is a set of finite sequences of 0's and 1's which is closed under initial segments. We say that a binary tree T has an infinite path if and only if there is an $f : \omega \to \{0,1\}$ such that, for every $n \in \omega$, $\langle f(0), \ldots, f(n-1)\rangle \in T$. Show that every infinite binary tree has an infinite path.

Proof. Let T be an infinite binary tree. Then T has an infinite number of nodes; that is, T contains an infinite number of finite sequences of 0's and 1's. This being the case, there must be either an infinite number of sequences having $\langle 0\rangle$ as an initial segment, or an infinite number of sequences having $\langle 1\rangle$ as an initial segment. Thus, let

$$f(0) = \begin{cases} 0 & \text{if there are an infinite number of elements of } T \text{ having} \\ & \langle 0\rangle \text{ as an initial segment} \\ 1 & \text{otherwise} \end{cases}$$

Let $T' = \{\text{sequences in } T \text{ having } \langle f(0)\rangle \text{ as an initial segment}\}$. Then T' is also an infinite tree. Therefore, there must be either an infinite number of sequences in T' having $\langle f(0), 0\rangle$ as an initial segment or an infinite number of sequences in T' having $\langle f(0), 1\rangle$ as an initial segment. Hence, we let

$$f(1) = \begin{cases} 0 & \text{if there are an infinite number of elements of } T' \text{ having} \\ & \langle f(0), 0\rangle \text{ as an initial segment} \\ 1 & \text{otherwise} \end{cases}$$

Let $T'' = \{\text{sequences in } T' \text{ having } (f(0), f(1) \text{ as an initial segment}\}$. Then T'' is also an infinite tree. Iterate this process.

At each stage, then, we generate $T^{(k)} \subset T$ which is an infinite tree in which each member has $\langle f(0), f(1), \ldots, f(k-1)\rangle$ as an initial segment. In such a way, we recursively define $f(n)$ for all $n \in \omega$.

Furthermore, given our definition of $f : \omega \to \{0,1\}$, we see that, for all $n \in \omega$,

$$\langle f(0), f(1), \ldots, f(n-1)\rangle \in T^{(n)} \subset T,$$

so T has an infinite path. □

4

Recall that PA is the extension of A_E by the first-order schema of mathematical induction. Show that if PA is consistent, then there is a consistent, but ω-inconsistent, theory S extending PA.

Proof. Let us list the sentences in the language of PA: $\phi_0, \phi_1, \ldots$ and all sequences in this language: $\sigma_0, \sigma_1, \ldots$. Let $\mathrm{Bew}(m,n)$ hold just in the case that σ_m is a proof of ϕ_n. By the fixed point lemma, there exists $n \in \omega$ such that

$$\mathsf{PA} \vDash \forall y \neg \mathrm{Bew}(y, \bar{n}) \leftrightarrow \phi_n.$$

Let $\Theta = \phi_n$. Suppose $\mathsf{PA} \vDash \Theta$. Then σ_m is a proof of $\Theta = \phi_n$ for some m, so

$$\mathsf{PA} \vDash \mathrm{Bew}(\bar{m}, \bar{n}).$$

Hence, $\mathsf{PA} \vDash \exists y \mathrm{Bew}(y, \bar{n})$. On the other hand, since $\mathsf{PA} \vDash \Theta$, $\mathsf{PA} \vDash \forall y \neg \mathrm{Bew}(y, \bar{n})$, which is equivalent to

$$\mathsf{PA} \vDash \neg \exists y \mathrm{Bew}(y, \bar{n}).$$

Therefore, if PA is consistent (as we are assuming it is), we see that $\mathsf{PA} \nvDash \Theta$. Therefore,

$$\begin{aligned} \mathsf{PA} &\vDash \neg \mathrm{Bew}(0, \bar{n}) \\ \mathsf{PA} &\vDash \neg \mathrm{Bew}(1, \bar{n}) \\ &\vdots \end{aligned}$$

Now, let $S = \mathsf{PA} \cup \{\Theta\}$. If S is inconsistent then (cf. Enderton Corollary 24E), $\mathsf{PA} \vDash \neg\Theta$. If so, then

$$\mathsf{PA} \vDash \neg \forall y \neg \mathrm{Bew}(y, \bar{n}),$$

which is to say

$$\mathsf{PA} \vDash \exists y \mathrm{Bew}(y, \bar{n}).$$

However, since $\mathsf{PA} \vDash \neg \mathrm{Bew}(\bar{m}, \bar{n})$ for all $m \in \omega$, since PA is consistent and since $\sigma_0, \sigma_1, \ldots$ is a complete list of sequences in the language of PA, we see that, in fact, $\mathsf{PA} \nvDash \exists y \mathrm{Bew}(y, \bar{n})$. Hence, $\mathsf{PA} \nvDash \neg\Theta$, so S is, in fact, consistent.

Now, clearly, $S \vDash \neg\Theta$; hence $S \vDash \neg \forall y \neg \mathrm{Bew}(y, \bar{n})$, which is equivalent to $S \vDash \exists y \neg\neg \mathrm{Bew}(y, \bar{n})$. On the other hand, since S is an extension of PA,

$$\begin{aligned} S &\vDash \neg \mathrm{Bew}(0, \bar{n}) \\ S &\vDash \neg \mathrm{Bew}(1, \bar{n}) \\ &\vdots \end{aligned}$$

Therefore, S is ω-inconsistent. $\square$