

GEOMETRY HW 10

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1

Show that $v_1, \dots, v_k \in V$ are linearly independent iff $v_1 \wedge \dots \wedge v_k \neq 0$ in $\Lambda_k(V)$.

Proof. Suppose v_1, \dots, v_k are linearly dependent. Then there exist $a_2, \dots, a_k \in \mathbb{R}$ such that

$$v_1 = a_2 v_2 + \dots + a_k v_k.$$

Then

$$\begin{aligned} v_1 \wedge \dots \wedge v_k &= (a_2 v_2 + \dots + a_k v_k) \wedge v_2 \wedge \dots \wedge v_k \\ &= a_2 v_2 \wedge v_2 \wedge \dots \wedge v_k + \dots + a_k v_k \wedge v_2 \wedge \dots \wedge v_k \\ &= 0 + \dots + 0 \\ &= 0 \end{aligned}$$

Hence, we conclude that if $v_1 \wedge \dots \wedge v_k \neq 0$, then v_1, \dots, v_k must be linearly independent.

On the other hand, suppose $v_1 \wedge \dots \wedge v_k = 0$. Then, for all linear maps $\phi : \Lambda_k(V) \rightarrow \mathbb{R}$, $\phi(v_1 \wedge \dots \wedge v_k) = 0$. Now, suppose $\psi : V \times \dots \times V \rightarrow \mathbb{R}$ is an alternating multilinear map. Then, by the universal property of the external product, ψ factors through a linear map $\Psi : \Lambda_k(V) \rightarrow \mathbb{R}$ such that $\psi = \Psi \circ \pi$, where $\pi : V \times \dots \times V \rightarrow \Lambda_k(V)$ is the standard projection. By the above argument, then, $\Psi(v_1 \wedge \dots \wedge v_k) = 0$, and so

$$\begin{aligned} \psi(v_1, \dots, v_k) &= \Psi \circ \pi(v_1, \dots, v_k) \\ &= \Psi(v_1 \wedge \dots \wedge v_k) \\ &= 0. \end{aligned}$$

Since our choice of ψ was arbitrary, we see that all alternating multilinear maps must be zero on (v_1, \dots, v_k) . In particular, if $\det_i : V \times \dots \times V \rightarrow \mathbb{R}$ is given by

$\det_i(w_1, \dots, w_k)$ = the determinant of the i th $k \times k$ minor of the matrix $[w_1 \dots w_k]$

then \det_i is an alternating multilinear map for all $i = 1, \dots, n - k$, and so $\det_i(v_1, \dots, v_k) = 0$ for all $i = 1, \dots, n - k$. This, then, implies that the matrix $[v_1 \dots v_k]$ has rank strictly less than k , and so its columns (namely the v_i) must be linearly dependent. Having proved the contrapositive, we conclude that if v_1, \dots, v_k are linearly independent, then $v_1 \wedge \dots \wedge v_k \neq 0$. \square

2

Recall that an element in $\Lambda_k(V)$ is decomposable iff it is of the form $v_1 \wedge \dots \wedge v_k$.

(a): Given four vectors $v, w, x, y \in V$, under what conditions is $v \wedge w + x \wedge y$ a decomposable vector in $\Lambda_2(V)$?

Answer: Let $\omega = v \wedge w + x \wedge y$. If ω is decomposable, then $\omega = z_1 \wedge z_2$ for some $z_1, z_2 \in V$; hence,

$$\omega \wedge \omega = (z_1 \wedge z_2) \wedge (z_1 \wedge z_2) = 0.$$

Hence,

$$\begin{aligned} 0 = \omega \wedge \omega &= (v \wedge w + x \wedge y) \wedge (v \wedge w + x \wedge y) \\ &= (v \wedge w) \wedge (v \wedge w) + (v \wedge w) \wedge (x \wedge y) + (x \wedge y) \wedge (v \wedge w) + (x \wedge y) \wedge (x \wedge y) \\ &= (v \wedge w) \wedge (x \wedge y) + (x \wedge y) \wedge (v \wedge w) \\ &= 2(v \wedge w \wedge x \wedge y), \end{aligned}$$

which means that $v \wedge w \wedge x \wedge y = 0$. By our work in problem 1 above, this implies that v, w, x, y are linearly dependent.

On the other hand, suppose v, w, x, y are linearly dependent. Then there exist a_1, a_2, a_3 such that

$$v = a_1 w + a_2 x + a_3 y.$$

Hence,

$$\begin{aligned} v \wedge w + x \wedge y &= (a_1 w + a_2 x + a_3 y) \wedge w + x \wedge y \\ &= a_1 w \wedge w + a_2 x \wedge w + a_3 y \wedge w + x \wedge y \\ &= a_2 x \wedge w + a_3 y \wedge w + x \wedge y \\ &= x \wedge a_2 w + y \wedge a_3 w + x \wedge y \\ &= x \wedge (a_2 w + y) + y \wedge a_3 w \\ &= x \wedge (a_2 w + y) + y \wedge a_3 w + (a_2 a_3) w \wedge w \\ &= x \wedge (a_2 w + y) + y \wedge a_3 w + a_2 w \wedge a_3 w \\ &= x \wedge (a_2 w + y) + (y + a_2 w) \wedge a_3 w \\ &= x \wedge (a_2 w + y) + a_3 w \wedge (a_2 w + y) \\ &= (x - a_3 w) \wedge (a_2 w + y); \end{aligned}$$

thus, $v \wedge w + x \wedge y$ is decomposable.

Therefore, we conclude that $v \wedge w + x \wedge y$ is decomposable if and only if v, w, x, y are linearly dependent. ♣

(b): Show that $\omega \in \Lambda_2(\mathbb{R}^4)$ is decomposable iff $\omega \wedge \omega = 0$.

Proof. Suppose, first of all, that ω is decomposable. Then there exist $v, w \in V$ such that $\omega = v \wedge w$. Then

$$\omega \wedge \omega = (v \wedge w) \wedge (v \wedge w) = -v \wedge v \wedge w \wedge w = 0$$

On the other hand, suppose $\omega \wedge \omega = 0$. Now, since $\omega \in \Lambda_2(\mathbb{R}^4)$, we know that ω can be expressed as a linear combination of basis

elements:

$$\omega = a_1(e_1 \wedge e_2) + a_2(e_1 \wedge e_3) + a_3(e_1 \wedge e_4) + a_4(e_2 \wedge e_3) + a_5(e_2 \wedge e_4) + a_6(e_3 \wedge e_4)$$

for some $a_1, \dots, a_6 \in \mathbb{R}$. Then this means that $a_i \neq 0$ for some i .

Hence,

$$\begin{aligned} 0 &= \omega \wedge \omega \\ &= (a_1(e_1 \wedge e_2) + a_2(e_1 \wedge e_3) + a_3(e_1 \wedge e_4) + a_4(e_2 \wedge e_3) + a_5(e_2 \wedge e_4) + a_6(e_3 \wedge e_4)) \\ &\quad \wedge (a_1(e_1 \wedge e_2) + a_2(e_1 \wedge e_3) + a_3(e_1 \wedge e_4) + a_4(e_2 \wedge e_3) + a_5(e_2 \wedge e_4) + a_6(e_3 \wedge e_4)) \\ &= (a_1(e_1 \wedge e_2)) \wedge (a_6(e_3 \wedge e_4)) + (a_2(e_1 \wedge e_3)) \wedge (a_5(e_2 \wedge e_4)) + (a_3(e_1 \wedge e_4)) \wedge (a_4(e_2 \wedge e_3)) \\ &\quad + (a_4(e_2 \wedge e_3)) \wedge (a_3(e_1 \wedge e_4)) + (a_5(e_2 \wedge e_4)) \wedge (a_2(e_1 \wedge e_3)) + (a_6(e_3 \wedge e_4)) \wedge (a_5(e_1 \wedge e_2)) \\ &= a_1 a_6 e_1 \wedge e_2 \wedge e_3 \wedge e_4 + a_2 a_5 e_1 \wedge e_3 \wedge e_2 \wedge e_4 + a_3 a_4 e_1 \wedge e_4 \wedge e_2 \wedge e_3 \\ &\quad + a_3 a_4 e_2 \wedge e_3 \wedge e_1 \wedge e_4 + a_2 a_5 e_2 \wedge e_4 \wedge e_1 \wedge e_3 + a_1 a_6 e_3 \wedge e_4 \wedge e_1 \wedge e_2 \\ &= 2a_1 a_6 e_1 \wedge e_2 \wedge e_3 \wedge e_4 - 2a_2 a_5 e_1 \wedge e_2 \wedge e_3 \wedge e_4 + 2a_3 a_4 e_1 \wedge e_2 \wedge e_3 \wedge e_4 \\ &= 2(a_1 a_6 + a_3 a_4 - a_2 a_5) e_1 \wedge e_2 \wedge e_3 \wedge e_4. \end{aligned}$$

Therefore, it must be the case that $a_1 a_6 + a_3 a_4 - a_2 a_5 = 0$, or

$$a_1 a_6 + a_3 a_4 = a_2 a_5.$$

Now, if it were the case that ω is decomposable, then we would have $\omega = v \wedge w$ for $v, w \in \mathbb{R}^4$. In turn, we can re-write v and w in terms of the standard basis of \mathbb{R}^4 :

$$\begin{aligned} v &= c_1 e_1 + c_2 e_2 + c_3 e_3 + c_4 e_4 \\ w &= d_1 e_1 + d_2 e_2 + d_3 e_3 + d_4 e_4 \end{aligned}$$

for some $c_1, c_2, c_3, c_4, d_1, d_2, d_3, d_4 \in \mathbb{R}$. In turn,

$$\begin{aligned} \omega = v \wedge w &= (c_1 e_1 + c_2 e_2 + c_3 e_3 + c_4 e_4) \wedge (d_1 e_1 + d_2 e_2 + d_3 e_3 + d_4 e_4) \\ &= c_1 d_2 e_1 \wedge e_2 + c_1 d_3 e_1 \wedge e_3 + c_1 d_4 e_1 \wedge e_4 + c_2 d_1 e_2 \wedge e_1 \\ &\quad + c_2 d_3 e_2 \wedge e_3 + c_2 d_4 e_2 \wedge e_4 + c_3 d_1 e_3 \wedge e_1 + c_3 d_2 e_3 \wedge e_2 \\ &\quad + c_3 d_4 e_3 \wedge e_4 + c_4 d_1 e_4 \wedge e_1 + c_4 d_2 e_4 \wedge e_2 + c_4 d_3 e_4 \wedge e_3 \\ &= (c_1 d_2 - c_2 d_1) e_1 \wedge e_2 + (c_1 d_3 - d_3 e_1) e_1 \wedge e_3 + (c_1 d_4 - c_4 d_1) e_1 \wedge e_4 \\ &\quad + (c_2 d_3 - c_3 d_2) e_2 \wedge e_3 + (c_2 d_4 - c_4 d_2) e_2 \wedge e_4 + (c_3 d_4 - c_4 d_3) e_3 \wedge e_4 \end{aligned}$$

Since we know $\omega \neq 0$, it must be the case that $a_i \neq 0$ for some i and $b_j \neq 0$ for some j . Suppose, without loss of generality, that $c_4 = d_4 = 1$ (we don't lose generality, because we can simply scale appropriately at any time in the below calculations). Hence, given these two descriptions of ω , we could write the system

$$\begin{aligned} a_1 &= c_1 d_2 - c_2 d_1 \\ a_2 &= c_1 d_3 - c_3 d_1 \\ a_3 &= c_1 d_4 - c_4 d_1 = c_1 - d_1 \\ a_4 &= c_2 d_3 - c_3 d_2 \\ a_5 &= c_2 d_4 - c_4 d_2 = c_2 - d_2 \\ a_6 &= c_3 d_4 - c_4 d_3 = c_3 - d_3. \end{aligned}$$

Then we see that

$$c_1 = a_3 + d_1 \quad c_2 = a_5 + d_2 \quad c_3 = a_6 + d_3.$$

Then

$$\begin{aligned}
a_1a_6 + a_3a_4 &= (c_1d_2 - c_2d_1)(c_3 - d_3) + (c_1 - d_1)(c_2d_3 - c_3d_2) \\
&= c_1c_3d_2 - c_2c_3d_1 - c_1d_2d_3 + c_2d_1d_3 + c_1c_2d_3 - c_1c_3d_3 - c_2d_1d_3 + c_3d_1d_2 \\
&= -c_2c_3d_1 - c_1d_2d_3 + c_1c_2d_3 + c_3d_1d_2 \\
&= c_1c_2d_3 - c_2c_3d_1 - c_1d_2d_3 + c_3d_1d_2 \\
&= (c_1d_3 - c_3d_1)(c_2 - d_2) \\
&= a_2a_5.
\end{aligned}$$

Therefore, we see that the above system of equations is consistent with having $\omega \wedge \omega = 0$. Thus, if we let $c_4 = d_4 = 1$ and let $c_1, c_2, c_3, d_1, d_2, d_3$ be solutions to the above system of equations, then, based solely on the fact that $\omega \wedge \omega = 0$, we see that

$$\omega = \left(\sum_{i=1}^4 c_i e_i \right) \wedge \left(\sum_{j=1}^4 d_j e_j \right),$$

so ω is decomposable. \square

3

Consider the definition of the wedge product of two alternating multilinear forms $f \in A_p(V)$, $g \in A_q(V)$ in Spivak:

$$(f \wedge g)(v_1, \dots, v_{p+q}) = \sum \text{sgn}(\sigma) f(v_{\sigma(1)}, \dots, v_{\sigma(p)}) g(v_{\sigma(p+1)}, \dots, v_{\sigma(p+q)})$$

where the sum is over all permutations σ with $\sigma(1) < \dots < \sigma(p)$ and $\sigma(p+1) < \dots < \sigma(p+q)$.

Show that this definition indeed follows, as claimed in class, from the natural God-given wedge product in $\Lambda(V^*)$ under the isomorphisms $A_k(V) \simeq \Lambda_k(V)^* \simeq \Lambda_k(V^*)$.

Proof. First, some notation. Call a permutation $\sigma \in S_{p+q}$ a *shuffle* if $\sigma(1) < \dots < \sigma(p)$ and $\sigma(p+1) < \dots < \sigma(p+q)$ and let $H \subset S_{p+q}$ be the set of all shuffles. Then, for each $\sigma \in S_{p+q}$, there corresponds a unique shuffle $\sigma' \in H$, namely that shuffle which is a re-arrangement of $\sigma(1), \dots, \sigma(p)$ into an ordering and $\sigma(p+1), \dots, \sigma(p+q)$ into an ordering. For example, if $p = 3$, $q = 2$ and $\sigma \in S_5$ such that

$$\begin{aligned}
\sigma(1) &= 1 \\
\sigma(2) &= 4 \\
\sigma(3) &= 3 \\
\sigma(4) &= 5 \\
\sigma(5) &= 2,
\end{aligned}$$

then σ' is that shuffle such that $\sigma'(1, 2, 3, 4, 5) = (1, 3, 4, 2, 5)$. This correspondence of each permutation with a unique shuffle forms a partition on S_{p+q} ; let σS_{p+q} denote the equivalence class of the shuffle σ .

Now, let $\omega \in \Lambda_p(V^*)$ be associated to f by the given isomorphism, and $\eta \in \Lambda_q(V^*)$ associated with g . Then

$$\omega = f_1 \wedge \dots \wedge f_p$$

and

$$\eta = g_1 \wedge \dots \wedge g_q$$

for $f_i, g_j : V^*$. Then, under the isomorphism,

$$f(v_1, \dots, v_p) = \det(f_i(v_j))_{ij} = \sum_{\sigma \in S_p} \text{sgn}(\sigma) \prod_{i=1}^p f_i(v_{\sigma(i)})$$

and

$$g(w_1, \dots, w_q) = \det(g_i(w_j))_{ij} = \sum_{\sigma \in S_q} \text{sgn}(\sigma) \prod_{i=1}^q g_i(v_{\sigma(i)}).$$

Therefore, the image of $\omega \wedge \eta$ in $A_{p+q}(V)$ is $f \wedge g$, where

$$\begin{aligned} (f \wedge g)(v_1, \dots, v_{p+q}) &= \det \left(\frac{f_i(v_j)}{g_i(v_{p+j})} \right) \\ &= \sum_{\sigma \in S_{p+q}} \text{sgn}(\sigma) \prod_{i=1}^p f_i(v_{\sigma(i)}) \prod_{j=1}^q g_j(v_{\sigma(p+j)}) \\ &= \sum_{\sigma \in H} \sum_{\tau \in \sigma S_{p+q}} \text{sgn}(\tau) \prod_{i=1}^p f_i(v_{\tau(i)}) \prod_{j=1}^q g_j(v_{\tau(p+j)}) \\ &= \sum_{\sigma \in H} \text{sgn}(\sigma) f(v_{\sigma(1)}, \dots, v_{\sigma(p)}) g(v_{\sigma(p+1)}, \dots, v_{\sigma(p+q)}), \end{aligned}$$

just as Spivak says. □

4

Consider the differential form on \mathbb{R}^3 :

$$\omega = zdy \wedge dx + xydx \wedge dz$$

and the map $f : \mathbb{R}^2 \rightarrow \mathbb{R}^3$ given by $f(x, y) = (x^2, x, xy)$.

Show that $d(f^*(\omega)) = f^*(d\omega)$ by explicitly computing both sides.

Proof. First, let us compute $f^*(\omega)$:

$$\begin{aligned} f^*(\omega) &= f^*(zdy \wedge dx + xydx \wedge dz) \\ &= f^*(zdy \wedge dx) + f^*(xydx \wedge dz) \\ &= xy(f^*(dy) \wedge f^*(dx)) + x^3(f^*(dx) \wedge f^*(dz)) \\ &= xy(dx \wedge (2xdx)) + x^3((2xdx) \wedge (ydx + xdy)) \\ &= 2x^2y(dx \wedge dx) + 2x^4y(dx \wedge dx) + 2x^5(dx \wedge dy) \\ &= 2x^5(dx \wedge dy) \end{aligned}$$

Hence,

$$d(f^*(\omega)) = (10x^4dx) \wedge dx \wedge dy = 0.$$

On the other hand,

$$\begin{aligned}
 d\omega &= dz \wedge dy \wedge dx + d(xy) \wedge dx \wedge dz \\
 &= dz \wedge dy \wedge dx + (ydx + xdy) \wedge dx \wedge dz \\
 &= -dx \wedge dy \wedge dz + y(dx \wedge dx \wedge dz) + x(dy \wedge dx \wedge dz) \\
 &= -dx \wedge dy \wedge dz - x(dx \wedge dy \wedge dz) \\
 &= (-1 - x)dx \wedge dy \wedge dz.
 \end{aligned}$$

Hence,

$$\begin{aligned}
 f^*(d\omega) &= f^*((-1 - x)dx \wedge dy \wedge dz) \\
 &= (-1 - x^2)(2xdx) \wedge dx \wedge (xdy + ydx) \\
 &= 0.
 \end{aligned}$$

Therefore, we conclude that, indeed, $d(f^*(\omega)) = f^*(d\omega)$. \square

5

Let ω be a k -form on N and $f : M \rightarrow N$ differentiable. Show that $f^*(\omega)$ is a differentiable k -form on M .

Proof. Suppose, first of all, that ω is decomposable. Then

$$\omega = gdx_1 \wedge \dots \wedge dx_k$$

for some smooth g . Then

$$f^*(\omega) = (g \circ f)df_1 \wedge \dots \wedge df_k$$

where the f_i are the i th coordinate functions. Then $f_i : M \rightarrow \mathbb{R}$, so $f_i \in \Omega^0(M)$ and, hence, $df_i \in \Omega^1(M)$ for all $i = 1, \dots, k$. Hence, $df_1 \wedge \dots \wedge df_k$ is a k -form. Furthermore, since both f and g are differentiable, so is $g \circ f$, and so

$$(g \circ f)df_1 \wedge \dots \wedge df_k$$

is a differentiable k -form on M .

More generally, if

$$\omega = \sum_{i=1}^m g_i v_{i_1} \wedge \dots \wedge v_{i_k},$$

then

$$f^*(\omega) = \sum_{i=1}^m (g_i \circ f)df_{i_1} \wedge \dots \wedge df_{i_k}$$

is a differentiable k -form, since each summand is by the argument given above in the decomposable case. \square

6

Show that the 2 form on $\mathbb{R}^2 - \{0\}$ given by

$$\omega = \frac{xdy - ydx}{x^2 + y^2}$$

is closed but not exact.

Proof. First, we show that ω is closed. To that end, let $f(x, y) = \frac{-y}{x^2+y^2}$ and $g(x, y) = \frac{x}{x^2+y^2}$ and re-write

$$\omega = \frac{-ydx}{x^2 + y^2} + \frac{xdy}{x^2 + y^2} = fdx + gdy.$$

Then

$$\begin{aligned} d\omega &= d(fdx + gdy) \\ &= df \wedge dx + dg \wedge dy \\ &= \left(\frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy \right) \wedge dx + \left(\frac{\partial g}{\partial x} dx + \frac{\partial g}{\partial y} dy \right) \wedge dy \\ &= \frac{\partial f}{\partial y} dy \wedge dx + \frac{\partial g}{\partial x} dx \wedge dy \\ &= \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right) dx \wedge dy. \end{aligned}$$

Now,

$$\frac{\partial f}{\partial y} = \frac{y^2 - x^2}{x^2 + y^2} = \frac{\partial g}{\partial x},$$

so we see that

$$d\omega = \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right) dx \wedge dy = 0,$$

so ω is closed.

On the other hand, suppose ω is exact. Then $\omega = d\eta$ for some $\eta \in \Omega^0(\mathbb{R}^2 - \{0\})$. Since the 0-forms are just smooth maps into \mathbb{R} , $\eta = h$ for some smooth $h : \mathbb{R}^2 - \{0\} \rightarrow \mathbb{R}$. Then

$$\omega = d\eta = dh = \frac{\partial h}{\partial x} dx + \frac{\partial h}{\partial y} dy.$$

Then, integrating, we see that

$$h = \int \frac{x}{x^2 + y^2} dy + g(x)$$

for some g and

$$h = \int \frac{-y}{x^2 + y^2} dx + f(y)$$

for some f . However, when we actually do the integrals, we see that

$$h(x, y) = \int \frac{x}{x^2 + y^2} dy + g(x) = \tan^{-1}(y/x) + g(x)$$

and

$$h(x, y) = \int \frac{-y}{x^2 + y^2} dx + f(y) = -\tan^{-1}(x/y) + f(y).$$

However, this is clearly impossible, so we conclude that there is, in fact, no such h and thus that ω is not exact. \square

7

Let G be a connected Lie group.

(a): Show that every discrete normal subgroup of G lies in the center of G .

Proof.] Let $H \subset G$ be a discrete normal subgroup. Let $h \in H$. Then, since multiplication and taking inverses are both continuous, the map $C_h : G \rightarrow G$ given by

$$g \mapsto ghg^{-1}$$

is continuous. Since G is connected, $C_h(G)$ is also connected. Now, since H is normal, $ghg^{-1} \in H$, so $C_h(G) \subset H$. Since H is discrete, the only connected subsets are single points, so $C_h(G)$ is a single point. Now, clearly, if e is the identity on G , $eh e^{-1} = h$, so we see that, in fact, $C_h(G) = \{h\}$. In turn, this means that, for all $g \in G$,

$$ghg^{-1} = h,$$

so $h \in Z(G)$. Since our choice of $h \in H$ was arbitrary, we see that this is true for all $h \in H$, so $H \subset Z(G)$. \square

(b): Show that the quotient of G by a discrete subgroup of the center is again a Lie group.

Proof. Let $H \subset Z(G)$ be discrete. Let $h \in H$. Then, if $hg = g$ for some $g \in G$, then $h = gg^{-1} = e$, so we see that the action of H on G is free.

Now, since H is discrete, there exists a neighborhood U of the identity e such that $U \cap H = \{e\}$. Let $V \subset U$ be a neighborhood of e such that $VV^{-1} \subset U$. Such a neighborhood certainly exists because multiplication and taking inverses is continuous in G . Now, let $h \in H$. If

$$hV \cap V \neq \emptyset,$$

then $hv = \tilde{v}$ for some $v, \tilde{v} \in V$. Since $VV^{-1} \subset U$,

$$h = \tilde{v}v^{-1} \in U.$$

Since $U \cap H = \{e\}$, we see that $h = e$. Hence, for $g \in G$, let $W = gV$ be a neighborhood of g . Then $hW \cap W \neq \emptyset$ only if $h = e$. Therefore, the action of H on G is free and properly discontinuous, so G/H is a manifold.

Note that G/H is also certainly a group, since H is normal in G . To see that multiplication and taking inverses are smooth on G/H , let $\pi : G \rightarrow G/H$ be the standard projection. Let $\bar{g}, \bar{g}' \in G/H$ and let $g \in \pi^{-1}(\bar{g})$, $g' \in \pi^{-1}(\bar{g}')$. Let (U, x) be a coordinate chart at g

and let (V, y) be a coordinate chart at $g'g$. Then $(\pi(U), x \circ \pi|_U^{-1})$ is a coordinate chart at \bar{g} and $(\pi(V), y \circ \pi|_V^{-1})$ is a coordinate chart at $\overline{g'g} = \overline{g'g}$. Then, since multiplication is continuous on G ,

$$y \circ L_{g'} \circ x^{-1} : x(U) \rightarrow y(V)$$

is a smooth map in \mathbb{R}^n . Hence,

$$y \circ L_{g'} \circ x^{-1} = y \circ (\pi|_V^{-1} \circ \pi|_V) \circ L_{g'} \circ (\pi|_U^{-1} \circ \pi|_U) \circ x^{-1} = (y \circ \pi|_V^{-1}) \circ (\pi|_V \circ L_{g'} \circ \pi|_U^{-1}) \circ (x \circ \pi|_U^{-1})^{-1}$$

is smooth as well. Now, for $\bar{a} \in \pi(U)$, if $a \in \pi^{-1}(\bar{a}) \cap U$, then

$$L_{\bar{g}} = \overline{g'a} = \pi(g'a) = \pi|_V \circ L_{g'} \circ \pi|_U^{-1}(a),$$

so we see that, in fact,

$$(y \circ \pi|_V^{-1}) \circ L_{\bar{g}} \circ (x \circ \pi|_U^{-1})^{-1}$$

is smooth, and so we conclude that left multiplication on G/H is smooth on G/H . A similar argument demonstrates that right multiplication is smooth on G/H .

Furthermore, if we let \bar{g}, g, U, x be as above and let (W, z) be a coordinate chart on g^{-1} , then, if we denote the map $g \mapsto g^{-1}$ on G by F and the map $\bar{g} \mapsto \bar{g}^{-1}$ on G/H by f , then

$$z \circ F \circ x^{-1}$$

is a smooth map from $x(U)$ to $z(W)$. Furthermore, since $g^{-1} \in \pi^{-1}(\bar{g}^{-1})$, $f = \pi|_W \circ F \circ \pi|_U^{-1}$ and so

$$\begin{aligned} z \circ F \circ x^{-1} &= z \circ (\pi|_W^{-1} \circ \pi|_W) \circ F \circ (\pi|_U^{-1} \circ \pi|_U) \circ x^{-1} \\ &= (z \circ \pi|_W^{-1}) \circ (\pi|_W \circ F \circ \pi|_U^{-1}) \circ (x \circ \pi|_U^{-1})^{-1} \\ &= (z \circ \pi|_W^{-1}) \circ f \circ (x \circ \pi|_U^{-1})^{-1} \end{aligned}$$

is smooth and, thus, f is smooth on G/H .

Therefore, since G/H is a manifold and a group and multiplication and taking inverses are smooth on G/H , we see that G/H is a Lie group. \square

(c): Let H be a closed Lie subgroup of G . Show that if H is normal in G , then G/H is again a Lie group.

Proof. Define a right action of H on G by

$$h \cdot g = gh^{-1}.$$

Then, if $h \cdot g = gh^{-1} = g$, for some $g \in G$, then $h = e$, so this action is free.

Now, let $K \subset G$ be compact. Consider $S = \{h \in H | hK \cap K \neq \emptyset\}$. Let $h \in S$. Then $hk = k'$ for some $k, k' \in K$. Hence, $h = k'k^{-1} \in KK^{-1}$. Hence, $h \in H \cap KK^{-1}$, so $S \subset H \cap KK^{-1}$. On the other hand, if $h \in H \cap KK^{-1}$, then $h = k'k^{-1}$ for some $k, k' \in K$. Hence, $hk = k'$, so $h \in S$ and thus $H \cap KK^{-1} \subset S$. Thus, since containment goes both ways, we see that $S = H \cap KK^{-1}$.

Since multiplication and taking inverses are smooth in G and the smooth image of a compact set is compact, KK^{-1} is compact since K is. Since H is closed and the intersection of a closed set with a compact set is compact, we see that $S = H \cap KK^{-1}$ is compact. Therefore, we see that the action of H on G is both free and proper, so G/H is a manifold and $\pi : G \rightarrow G/H$ is a submersion. Furthermore, since $H \subset G$ is normal, G/H is a group.

Now, let $\bar{g}, \bar{g}' \in G/H$ and $g \in \pi^{-1}(\bar{g})$, $g' \in \pi^{-1}(\bar{g}')$. Let (U, x) be a coordinate chart on g and (V, y) a coordinate chart on g' . Then

$$\begin{array}{ccc} G \times G & \xrightarrow{L_g} & G \\ \pi \times \pi \downarrow & & \downarrow \pi \\ G/H \times G/H & \xrightarrow{L_{\bar{g}}} & G/H \end{array}$$

is a commutative diagram; i.e.,

$$L_{\bar{g}} \circ (\pi \times \pi) = \pi \circ L_g.$$

Since $\pi \times \pi$, π and L_g are all smooth, this implies that $L_{\bar{g}}$ is smooth. A similar argument shows that right multiplication is smooth on G/H .

Now, let \bar{g}, g, U, x . If $F : G \rightarrow G$ is the inverse map on G and $f : G/H \rightarrow G/H$ is the inverse map on G/H , then

$$\begin{array}{ccc} G & \xrightarrow{F} & G \\ \pi \downarrow & & \downarrow \pi \\ G/H & \xrightarrow{f} & G/H \end{array}$$

is a commutative diagram; i.e.,

$$f \circ \pi = \pi \circ F.$$

Since π and F are smooth, this implies that f is smooth.

Therefore, since G/H is both a group and a manifold and multiplication and taking inverses are smooth on G/H , we see that G/H is a Lie group. \square