GEOMETRY HW 12

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1

Show that the Stokes Theorem of vector calculus is a special case of the Stokes Theorem for manifolds.

Proof. Stokes' Theorem from vector calculus states that, if F is a vector field, $c : [0,1] \to \mathbb{R}^3$ a closed curve bounding a surface S and n a normal vector to the surface, then

$$\int_c F \cdot dr = \int_S \text{curl } F \cdot n dS$$

Suppose $F = P\frac{\partial}{\partial x} + Q\frac{\partial}{\partial y} + R\frac{\partial}{\partial z}$. Then, as we showed on problem 1 of the last homework, if $\omega = Pdx + Qdy + Rdz$, then

$$d\omega = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right) dy \wedge dz + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right) dz \wedge dx + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx \wedge dy,$$

which is equal to curl F under the identifications

$$\begin{aligned} dy \wedge dz &\rightsquigarrow \frac{\partial}{\partial x} \\ dz \wedge dx &\rightsquigarrow \frac{\partial}{\partial y} \\ dx \wedge dy &\rightsquigarrow \frac{\partial}{\partial z}. \end{aligned}$$

Now, $F \cdot dr = F \cdot Tds$ for tangent vector T. Then at a point $c(s) = (c_1(s), c_2(s), c_3(s))$ on the curve, $T = c'(s) = (c_1'(s), c_2'(s), c_3'(s))$, so

$$F \cdot dr = F \cdot Tds = (P(c(s))c_1'(s) + Q(c_2(s))c_2'(s) + R(c_3(s))c_3'(s))ds = c^*(\omega).$$

Hence,

$$\int_c \omega = \int_{[0,1]} c^*\omega = \int_0^1 F \cdot Tds = \int_c F \cdot dr$$

On the other hand, as we proved in class while proving the divergence theorem,

$$ndS = dy \wedge dz + dz \wedge dx + dx \wedge dy,$$

so

$$\text{curl } F \cdot ndS = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right) dy \wedge dz + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right) dz \wedge dx + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx \wedge dy = d\omega.$$

Hence,

$$\int_S F \cdot ndS = \int_S d\omega.$$

Therefore, by Stokes' Theorem on manifolds,

$$\int_S F \cdot ndS = \int_S d\omega = \int_c \omega = \int_c F \cdot dr.$$

□

2

Show that the wedge product for forms induces a product in De Rham cohomology. This product is called the cup product and denoted by $\alpha \cup \beta$. State some of the properties of this cup product.

Proof. Define the cup product $\cup : H^k(M) \times H^l(M) \to H^{k+l}(M)$ by

$$[\omega] \cup [\eta] = [\omega \wedge \eta].$$

Suppose ω and η are closed. If $\omega = d\gamma$ for some $\gamma \in \Omega^{k-1}(M)$, then

$$d(\gamma \wedge \eta) = d\gamma \wedge \eta + \gamma \wedge d\eta = d\gamma \wedge \eta = \omega \wedge \eta,$$

so $\omega \wedge \eta$ is exact. Similarly, if $\eta = d\nu$ for some $\nu \in \Omega^{l-1}(M)$, then

$$d(\omega \wedge \nu) = d\omega \wedge \nu + \omega \wedge d\nu = \omega \wedge d\nu = \omega \wedge \eta,$$

so $\omega \wedge \eta$ is exact.

Now, suppose $\omega' \in [\omega]$ and $\eta' \in [\eta]$ for $[\omega] \in H^k(M)$ and $[\eta] \in H^l(M)$. Then $\omega' = \omega + \xi$ for some exact form ξ and $\eta' = \eta + \zeta$ for some exact for ζ. Hence

$$\begin{aligned} \omega' \wedge \eta' - \omega \wedge \eta &= (\omega + \xi) \wedge (\eta + \zeta) - \omega \wedge \eta \\ &= \omega \wedge \eta + \xi \wedge \eta + \omega \wedge \zeta + \xi \wedge \zeta - \omega \wedge \eta \\ &= \xi \wedge \eta + \omega \wedge \zeta + \xi \wedge \zeta. \end{aligned}$$

However, by what we showed above, since ξ and ζ are exact, each of the terms in this sum are exact, so the entire sum is exact and, hence, $[\omega' \wedge \eta'] = [\omega \wedge \eta]$. Therefore, the cup product defined above is well-defined.

Now, suppose $[\omega], [\omega'] \in H^k(M)$ and $[\eta] \in H^l(M)$ and $a, b \in \mathbb{R}$. Then

$$\begin{aligned} (a[\omega] + b[\omega']) \cup [\eta] = [a\omega + b\omega'] \cup [\eta] = [(a\omega + b\omega') \wedge \eta] &= [a\omega \wedge \eta + b\omega' \wedge \eta] \\ &= a[\omega \wedge \eta] + b[\omega' \wedge \eta] \\ &= a([\omega] \cup [\eta]) + b([\omega'] \cup [\eta]), \end{aligned}$$

so $\cup$ is linear in the first term. A similar calculation shows that $\cup$ is linear in the second term, so we conclude that $\cup$ is bilinear.

Now, suppose $[\omega] \in H^k(M)$ and $[\eta] \in H^l(M)$. Then $\omega = \sum_{i=1}^k f_i dx^{i_1} \wedge \ldots \wedge dx^{i_k}$ and $\eta = \sum_{j=1}^l g_j dx^{j_1} \wedge \ldots \wedge dx^{j_l}$. Hence

$$\begin{aligned}
[\eta] \cup [\omega] &= [\eta \wedge \omega] \\
&= \left[\left(\sum_{j=1}^{l} g_j dx^{j_1} \wedge \ldots \wedge dx^{j_l}\right) \wedge \left(\sum_{i=1}^{k} f_i dx^{i_1} \wedge \ldots \wedge dx^{i_k}\right)\right] \\
&= \left[\sum_{j=1}^{l}\sum_{i=1}^{k} g_j f_i dx^{j_1} \wedge \ldots \wedge dx^{j_l} \wedge dx^{i_1} \wedge \ldots \wedge dx^{i_k}\right] \\
&= \left[\sum_{i=1}^{k}\sum_{j=1}^{l} f_i g_j (-1)^{kl} dx^{i_1} \wedge \ldots \wedge dx^{i_k} \wedge dx^{j_1} \wedge \ldots \wedge dx^{j_l}\right] \\
&= (-1)^{kl}\left[\left(\sum_{i=1}^{k} f_i dx^{i_1} \wedge \ldots \wedge dx^{i_k}\right) \wedge \left(\sum_{j=1}^{l} g_j dx^{j_1} \wedge \ldots \wedge dx^{j_l}\right)\right] \\
&= (-1)^{kl}[\omega \wedge \eta].
\end{aligned}$$

□

3

Show that $H^1_{DR}(S^1) = \mathbb{R}$ by "bare hands", i.e. not quoting any big theorems in Spivak, except Stokes' Theorem.

Proof. Define $I : H^1(S^1) \to \mathbb{R}$ by

$$I([\omega]) = \int_{S^1} \omega.$$

First, if $\omega \in \Omega^1(S^1)$ is closed and $\omega' = \omega + \eta$ for some exact form $\eta = df$, note that

$$I([\omega']) = \int_{S^1} \omega' = \int_{S^1} \omega + \eta = \int_{S^1} \omega + \int_{S^1} \eta = \int_{S^1} \omega + \int_{\partial S^1} f = \int_{S^1} \omega,$$

since S^1 is a compact manifold without boundary, so I is well-defined.

Now, suppose that $[\omega], [\eta] \in H^1(S^1)$ and $a, b \in \mathbb{R}$. Then

$$I(a[\omega]+b[\eta]) = I([a\omega+b\eta]) = \int_{S^1} a\omega+b\eta = a\int_{S^1} \omega+b\int_{S^1} \eta = aI([\omega])+bI([\eta]),$$

so we see that I is linear.

Now, suppose $[\omega] \in H^1(S^1)$ such that $I([\omega]) = 0$. Let $\gamma : S^1 \to S^1$ be a closed curve. Then γ wraps around S^1 n times for some integer n. γ may also backtrack on itself, but this backtracking has no impact on the integral of ω along γ, since traversing the circle in the opposite direction only changes the sign of the integral. Hence,

$$\int_\gamma \omega = n\int_{S^1}\omega = I([\omega]) = 0.$$

Since our choice of γ was arbitrary, we see that $\int_\gamma \omega = 0$ for all closed curves γ; since S^1 is compact, orientable, has no boundary and is connected, our result proved in problem 4 below demonstrates that ω is exact, and so $[\omega] = 0$. Since our only restriction on ω was that $I([\omega]) = 0$, we see that ker $I = 0$. Therefore, since I is linear, this implies that $H^1(S^1)$ has dimension either 0 or 1.

Now, let $\omega = \frac{xdy - ydx}{x^2+y^2}$. Let $f(x,y) = \frac{-y}{x^2+y^2}$ and $g(x,y) = \frac{y}{x^2+y^2}$. Then

$$\begin{aligned}
\int_{S^1}\omega &= \int_{[0,1]} c^*(\omega)\\
&= \int_{[0,1]} (f\circ c)dc_1 + (g\circ c)dc_2\\
&= \int_{[0,1]} \left(\frac{-\sin 2\pi t}{\cos^2 2\pi t + \sin^2 2\pi t}(-2\pi\sin 2\pi t) + \frac{\cos 2\pi t}{\cos^2 2\pi t + \sin^2 2\pi t}2\pi\cos 2\pi t\right) dt\\
&= 2\pi\int_{[0,1]} \sin^2 2\pi t + \cos^2 2\pi t dt\\
&= 2\pi\int_{[0,1]} dt\\
&= 2\pi\\
&\neq 0,
\end{aligned}$$

so we see that there are non-zero elements of $H^1(S^1)$. Therefore, we conclude that, since $I : H^1(S^1) \to \mathbb{R}$ is injective, I is an isomorphism of vector spaces. □

4

Let M be compact, oriented, with no boundary, connected. Let ω be a closed one form on M such that $\int_\gamma \omega = 0$ for every closed curve γ. Show that ω is exact.

Proof. Fix $p_0 \in M$. Define the map $g : M \to \mathbb{R}$ by

$$g(p) = \int_{\gamma_p}\omega$$

where $\gamma_p : [0,1] \to M$ is a smooth curve from p_0 to p. To see that this is well-defined, let γ_p^1 and γ_p^2 be two smooth curves from p_0 to p. Define $\gamma : [0,2] \to M$ by

$$\gamma(t) = \begin{cases} \gamma_p^1(t) & 0 \leq t \leq 1\\ \gamma_p^2(2-t) & 1 \leq t \leq 2\end{cases}$$

Then, by the pasting lemma, γ is a smooth curve, since

$$\gamma(1) = \gamma_p^1(1) = \gamma_p^2(1).$$

Furthermore,

$$\gamma(0) = \gamma_p^1(0) = p_0 = \gamma_p^2(0) = \gamma(2),$$

so γ is a closed curve. Then

$$0 = \int_\gamma \omega = \int_{\gamma_p^1} \omega - \int_{\gamma_p^2} \omega,$$

so $\int_{\gamma_p^1} \omega = \int_{\gamma_p^2} \omega$. Since our choice of γ_p^1 and γ_p^2 were arbitrary, we see that this holds for all γ_p from p_0 to p, and so g is well-defined.

Now, let $p \in M$ and let (U, x) be a coordinate chart containing p. Then, since ω is a 1-form,

$$\omega = \sum_{i=1}^n f_i dx^i$$

in U. Let α_t be a path from $p + tx^i$ to p for some $i \in \{1, \ldots, n\}$. Then, since the integral is independent of path,

$$\int_{\gamma_p} \omega = \int_{\gamma_{p+tx^i}} \omega + \int_\alpha \omega$$

since γ_{p+tx^i} followed by α is a path from p_0 to p. Thus

$$\int_{\gamma_{p+tx^i}} \omega - \int_{\gamma_p} \omega = \int_{\gamma_{p+tx^i}} \omega - \left(\int_{\gamma_{p+tx^i}} \omega - \int_\alpha \omega \right) = \int_\alpha \omega$$

and, hence,

$$\begin{aligned} \left. \frac{\partial g}{\partial x^i} \right|_p &= \frac{d}{dt} \left[g(p + tx^i) - g(p) \right] \\ &= \frac{d}{dt} \left[\int_{\gamma_{p+tx^i}} \omega - \int_{\gamma_p} \omega \right] \\ &= \frac{d}{dt} \left[\int_\alpha \omega \right] \\ &= \frac{d}{dt} \left[\int_\alpha x^*(\omega) \right] \\ &= \frac{d}{dt} \left[\int_0^t f_i dx^i \right] \\ &= f_i \end{aligned}$$

by the fundamental theorem of calculus. Therefore, since f_i is smooth for all i, g is smooth and, furthermore,

$$dg = \sum_{i=1}^n \frac{\partial g}{\partial x^i} dx^i = \sum_{i=1}^n f_i dx^i = \omega,$$

so we see that ω is exact. $\square$

5

Show that $H^1_{DR}(T^2) = \mathbb{R}^2$ as follows: Let $T^2 = \mathbb{R}^2/(x, y) \mapsto (x+n, y+m)$. Then show that dx and dy become well defined forms on T^2 which form a basis of $H^1_{DR}(T^2)$.

Proof. To see that dx and dy are well-defined forms, note that

$$d(x + n) = dx + dn = dx + 0 = dx$$

and

$$d(y + m) = dy + dm = dy + 0 = dy.$$

Now, let $\alpha : [0, 1] \to \mathbb{R}^2$ be given by

$$t \mapsto (t, 0)$$

and $\beta : [0, 1] \to \mathbb{R}^2$ be given by

$$t \mapsto (0, t).$$

Then α and β are smooth curves on $\mathbb{R}^2$ and so define smooth curves in the quotient T^2. Furthermore, in T^2,

$$\alpha(0) = [(0, 0)] = [(1, 0)] = \alpha(1)$$

and

$$\beta(0) = [(0, 0)] = [(0, 1)] = \beta(1),$$

so α and β are closed curves in T^2. Now,

$$\int_\alpha dx = \int_{[0,1]} \alpha^*(dx) = \int_{[0,1]} (1 \circ \alpha) d\alpha_1 = \int_{[0,1]} tdt = 1$$

and

$$\int_\beta dy = \int_{[0,1]} \beta^*(dy) = \int_{[0,1]} (1 \circ \beta) d\beta_2 = \int_{[0,1]} tdt = 1.$$

On the other hand, if dx were exact, then we would have $dx = df$ for some smooth f and

$$1 = \int_\alpha dx = \int_{\partial\alpha} f = \int_0 f = 0,$$

so we see that dx is not exact. A similar argument shows that dy is not exact. On the other hand,

$$d(dx) = d(1) \wedge dx = 0 \wedge dx = 0$$

and

$$d(dy) = d(1) \wedge dy = 0 \wedge dy = 0,$$

so we see that dx and dy are closed. Hence, $[dx]$ and $[dy]$ are non-zero elements of $H^1(T^2)$.

Suppose that $[dx]$ and $[dy]$ are not linearly independent. Then there exist $c_1, c_2 \in \mathbb{R}$ such that

$$c_1[dx] + c_2[dy] = 0,$$

or $[dx] = \frac{c_2}{c_1}[dy]$; let $c = \frac{c_2}{c_1}$, so $dx = cdy + \eta$ for some exact form η. However, this implies that

$$c = c\int_\beta dy = \int_\beta cdy = \int_\beta dx - \eta = \int_{[0,1]} \beta^*(dx) - \int_\beta \eta = \int_{[0,1]} (1\circ\beta) d\beta_1 = \int_{[0,1]} 0 = 0.$$

Since $[dx] \neq 0$, this is impossible, so we conclude that $[dx], [dy]$ are linearly independent in $H^1(T^2)$.

Now, let $[\omega] \in H^1(T^2)$. Then

$$\int_\alpha \omega = c_1$$

and

$$\int_\beta \omega = c_2$$

for some $c_1, c_2 \in \mathbb{R}$. Let γ be a closed curve in T^2. Then γ is homotopy equivalent to a path δ where δ is simply a path that iterates α n times for some $n \in \mathbb{Z}$ and follows that with m iterations of β for some $m \in \mathbb{Z}$. Then

$$\begin{aligned}
\int_\gamma \omega - c_1 dx - c_2 dy &= \int_\gamma \omega - \int_\gamma c_1 dx - \int_\gamma c_2 dy \\
&= n\int_\alpha \omega + m\int_\beta \omega - \left(n\int_\alpha c_1 dx + m\int_\beta c_1 dx\right) - \left(n\int_\alpha c_2 dy + m\int_\beta c_2 dy\right) \\
&= nc_1 + mc_2 - nc_1\int_\alpha dx - mc_2\int_\beta dy \\
&= nc_1 + mc_2 - nc_1 - mc_2 \\
&= 0.
\end{aligned}$$

Since our choice of closed path γ was arbitrary, we see that

$$\int_\gamma \omega - (c_1 dx + c_2 dy) = 0$$

for all closed paths γ and so, by problem 4 above, $\omega - (c_1 dx + c_2 dy)$ is exact. Therefore,

$$[\omega] = [c_1 dx + c_2 dy] = c_1[dx] + c_2[dy].$$

Since our choice of $[\omega] \in H^1(T^2)$ was arbitrary, we see that $[dx]$ and $[dy]$ generate $H^1(T^2)$; since they are linearly independent, they form a basis of $H^1(T^2)$, and so the map

$$c_1[dx] + c_2[dy] \mapsto (c_1, c_2)$$

is an isomorphism of $H^1(T^2)$ with $\mathbb{R}^2$. □

6

Show that $H^1_{DR}(S^2) = 0$ and conclude that T^2 and S^2 cannot be diffeomorphic or even homotopy equivalent.

Proof. Let $\gamma : [0,1] \to S^2$ be a simple closed curve. Then the region enclosed by γ in M (call it R) is a manifold with boundary. Hence, by Stokes' Theorem, if ω is a closed 1-form on S^2,

$$0 = \int_R 0 = \int_R d\omega = \int_\gamma \omega,$$

so we see that $\int_\gamma \omega = 0$ for every simple closed curve γ.

Now, let $\gamma : [0,1] \to S^2$ be any closed curve on S^2. Then γ can be smoothly contracted down to the base point $\gamma(0) = \gamma(1)$, so γ is homotopic to the constant loop based at $\gamma(0)$, which in turn is homotopic to a simple closed curve based at $\gamma(0)$; call this simple closed curve β. Then, by the above result,

$$0 = \int_\beta \omega = \int_{[0,1]} \beta^*(\omega) = \int_{[0,1]} \gamma^*(\omega) = \int_\gamma \omega$$

so we see that $\int_\gamma \omega = 0$ for all closed curves γ. Hence, since S^2 is compact, oriented, has no boundary and is connected, our result in problem 4 above implies that ω is exact.

Therefore, all closed 1-forms on S^2 are exact, and so $H^1(S^2) = 0$. Therefore, since diffeomorphic (and even homotopy equivalent) manifolds have the same DeRham cohomology, we see that S^2 and T^2 are not diffeomorphic (or even homotopy equivalent), since

$$H^1(T^2) = \mathbb{R}^2 \neq 0 = H^1(S^2).$$

□

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