GEOMETRY HW 7

CLAY SHONKWILER

1

Let K be the Klein bottle defined as the quotient of $\mathbb{R}^2$ by $(x, y) \to (x, y+1)$ and $(x, y) \to (x+1, -y)$. Show that the span of $\frac{\partial}{\partial x}$ and the span of $\frac{\partial}{\partial y}$ define two smooth 1-dimensional distributions on K. Find their maximal integral manifolds and determine their lengths.

Answer: Let Δ_1 denote the span of $\frac{\partial}{\partial x}$ and Δ_2 the span of $\frac{\partial}{\partial y}$. Obviously, for each $p \in K$, $\Delta_{i_p} \subset T_pK$ is a 1-dimensional subspace of T_pK. Also, for any neighborhood U of p, if we let $X_1(q) = \frac{\partial}{\partial x}|_q$ and $X_2(q) = \frac{\partial}{\partial y}|_q$, then X_1 and X_2 are (locally) smooth vector fields on U and, for all $q \in U$,

$$\Delta_{i_q} = \mathrm{Span}\{X_i(q)\}.$$

Hence, Δ_i is a smooth 1-dimensional distribution on K for $i = 1, 2$. Now, since Δ_i is 1-dimensional, the associated integral manifolds will also be 1-dimensional; that is to say, simply integral curves. Now, if c is an integral curve through the point (x_0, y_0) associated with Δ_1, then $c(t) = (c_1(t), c_2(t))$ and

$$(c_1'(t), c_2'(t)) = c'(t) = a\frac{\partial}{\partial x} \circ c(t) = a\frac{\partial}{\partial x} = (a, 0)$$

for some $a \in \mathbb{R}$ (since $c'(t) \in \Delta_{1_{(x_0,y_0)}}$); hence, $c_2(t)$ is a constant and $c_1(t) = at + \kappa_1$ for some constant κ_1. Using the fact that $c(0) = (x_0, y_0)$, we see that $c_1(t) = at + x_0$ and $c_2(t) = y_0$, or

$$c(t) = (at + x_0, y_0).$$

Since c is defined on the entire line, this is the maximal integral curve through the point (x_0, y_0) associated with Δ_1; geometrically, c is just a horizontal line:

On the other hand, if d is an integral curve through the point (x_0, y_0) associated with Δ_2, then $d(t) = (d_1(t), d_2(t))$ and

$$(d_1'(t), d_2'(t)) = d'(t) = b\frac{\partial}{\partial y} \circ d(t) = b\frac{\partial}{\partial y} = (0, b)$$

for some $b \in \mathbb{R}$; hence, $d_1(t)$ is constant and $d_2(t) = bt + \kappa_2$ for some constant κ_2. Using our initial condition $d(0) = (x_0, y_0)$, we see that $d_1(t) = x_0$ and $d_2(t) = bt + y_0$ or

$$d(t) = (x_0, bt + y_0).$$

Since d is defined on the entire line, this is the maximal integral manifold through (x_0, y_0) associated with Δ_2; geometrically, d is just the vertical line:

♣

2

Which one of the following distributions on $\mathbb{R}^3$ are completely integrable:

(a): Δ spanned by $z\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}$ and $\frac{\partial}{\partial x} + x\frac{\partial}{\partial z}$.

(b): Δ spanned by $z\frac{\partial}{\partial x} + yz\frac{\partial}{\partial z}$ and $2y\frac{\partial}{\partial y} + 2xy\frac{\partial}{\partial z}$.

(c): Δ spanned by $y\frac{\partial}{\partial x} + x\frac{\partial}{\partial z}$.

If the distribution is completely integrable, find its maximal integral manifolds.

Answer: The distribution given in (a) is not integrable, as $X = z\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}$ and $Y = \frac{\partial}{\partial x} + x\frac{\partial}{\partial z}$ are certainly in Δ, but

$$\begin{aligned} [X, Y] &= [z\tfrac{\partial}{\partial x} + x\tfrac{\partial}{\partial y}, \tfrac{\partial}{\partial x} + x\tfrac{\partial}{\partial z}] \\ &= z\tfrac{\partial}{\partial z} - (\tfrac{\partial}{\partial y} + x\tfrac{\partial}{\partial x}) \\ &= -x\tfrac{\partial}{\partial x} - \tfrac{\partial}{\partial y} + z\tfrac{\partial}{\partial z}. \end{aligned}$$

However, if $[X, Y] \in \Delta$, then there exist scalars a, b such that

$$a(z\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}) + b(\frac{\partial}{\partial x} + x\frac{\partial}{\partial z}) = -x\frac{\partial}{\partial x} - \frac{\partial}{\partial y} + z\frac{\partial}{\partial z}.$$

This, in turn, implies that

$$\begin{aligned} a(z+1) &= -x \\ ax &= -1 \\ bx &= z. \end{aligned}$$

This implies $a = \frac{-1}{x}$, and so $x = \frac{z}{x} - b$ (from the second and first equations, respectively). Hence,

$$b = \frac{z - x^2}{x}.$$

Thus, by the third equation,

$$z = bx = \frac{z - x^2}{x} x = z - x^2;$$

since x is not always zero, this is clearly impossible, so we see that the distribution given in (a) is not completely integrable.

The distribution given in (b) is also not integrable; to see why, let $X = z\frac{\partial}{\partial x} + yz\frac{\partial}{\partial z}$ and $Y = 2y\frac{\partial}{\partial y} + 2xy\frac{\partial}{\partial z}$. Clearly, $X, Y \in \Delta$. However,

$$\begin{aligned}[X,Y] &= [z\tfrac{\partial}{\partial x} + yz\tfrac{\partial}{\partial z}, 2y\tfrac{\partial}{\partial y} + 2xy\tfrac{\partial}{\partial z}] \\ &= 2zy\tfrac{\partial}{\partial z} - (2yz\tfrac{\partial}{\partial z} + 2xy\tfrac{\partial}{\partial x} + 2xy^2\tfrac{\partial}{\partial z}) \\ &= -2xy\tfrac{\partial}{\partial x} - 2xy^2\tfrac{\partial}{\partial z}.\end{aligned}$$

However, if $[X, Y] \in \Delta$, then there exist a, b such that

$$a(z\frac{\partial}{\partial x} + yz\frac{\partial}{\partial z}) + b(2y\frac{\partial}{\partial y} + 2xy\frac{\partial}{\partial z}) = -2xy\frac{\partial}{\partial x} - 2xy^2\frac{\partial}{\partial z}.$$

Thus,

$$\begin{aligned} az &= -2xy \\ 2by &= 0 \\ ayz + 2bxy &= -2xy^2.\end{aligned}$$

Hence, $b = 0$ and $a = \frac{-2xy}{z}$. However, this is certainly not defined when $z = 0$ and, in fact, there is no solution of, say, the first equation when $z = 0$ and x, y non-zero. Hence, we see that the distribution defined in (b), although it is integrable on most of $\mathbb{R}^3$, is not integrable on all of $\mathbb{R}^3$ and, thus, not completely integrable.

The distribution defined in (c) is completely integrable. To see why, note that, if $X, Y \in \Delta$, then $X = a(y\frac{\partial}{\partial x} + x\frac{\partial}{\partial z})$ and $Y = b(y\frac{\partial}{\partial x} + x\frac{\partial}{\partial z})$ for some scalars a, b. Then

$$\begin{aligned}[X,Y] &= [a(y\tfrac{\partial}{\partial x} + x\tfrac{\partial}{\partial z}), b(y\tfrac{\partial}{\partial x} + x\tfrac{\partial}{\partial z})] \\ &= aby\tfrac{\partial}{\partial z} - aby\tfrac{\partial}{\partial z} \\ &= 0,\end{aligned}$$

which is certainly in Δ. Since our choice of X and Y was arbitrary, we see that this holds true for all vector fields X and Y contained in the distribution, so the distribution is completely integrable.

Since Δ is a 1-dimensional distribution, its maximal integral manifolds will also be 1-dimensional. Since a 1-dimensional integral manifold is just an integral curve, we can simply use our familiar method for finding integral curves. Then, if $c(t)$ is an integral curve through the point (x_0, y_0, z_0),

$$c(t) = (c_1(t), c_2(t), c_3(t))$$

and

$$(c_1'(t), c_2'(t), c_3'(t)) = c'(t) = c_2(t)\frac{\partial}{\partial x} + c_1(t)\frac{\partial}{\partial z} = (c_2(t), 0, c_1(t)).$$

That is,

$$\begin{aligned} c_1'(t) &= c_2(t) \\ c_2'(t) &= 0 \\ c_3'(t) &= c_1(t). \end{aligned}$$

So c_2 is constant; since $c(0) = (x_0, y_0, z_0)$, we see that $c_2(t) = y_0$. Thus,

$$c_1(t) = \frac{y_0}{2}t^2 + d$$

and

$$c_3(t) = \frac{y_0}{6}t^3 + dt + e$$

for scalars d and e. By the initial conditions,

$$x_0 = c_1(0) = d$$

and

$$z_0 = c_3(0) = e,$$

so we conclude that the integral manifold through (x_0, y_0, z_0) is given by

$$c(t) = (\frac{y_0}{2}t^2 + x_0, y_0, \frac{y_0}{6}t^3 + x_0 t + z_0).$$

Since this curve is defined for all $t \in \mathbb{R}$, we see that it is already maximal.

♣

3

Let $f : M^n \to N^{n-k}$ be a smooth map with no critical points. Show that the distribution $\Delta_p = \{v \in T_pM | D(f)_p(v) = 0\}$ is involutive and determine the maximal integrable manifolds.

Proof. Note, first, that since f has no critical points, rank $f = n - k$ at all points in M. Hence, Δ is $n - (n-k) = k$-dimensional. Now, suppose $X, Y \in \Delta$. Then

$$D(f)_p(X_p) = 0 = D(f)_p(Y_p),$$

by definition. Thus,

$$\begin{aligned} D(f)_p([X,Y](p)) &= [X,Y]_p(f) \\ &= (XY)_p(f) - (YX)_p(f) \\ &= X(Y(f)) - Y(X(f)) \\ &= X(D(f)_p(Y_p)) - Y(D(f)_p(X_p)) \\ &= X(0) - Y(0) \\ &= 0 \end{aligned}$$

which is certainly contained in Δ_p. Hence, Δ is involutive.

Now, let $p \in M$ and let U be a maximal neighborhood of p such that there exist (local) vector fields $x_1, \ldots, x_k$ such that

$$\langle x_1(q), \ldots, x_k(q) \rangle = \Delta_q$$

for all $q \in U$. Specifically, note that $x_i(q) \in \Delta_q$ for all $i = 1, \ldots, k$, all $q \in U$. Since the x_i are linearly independent, there exists some maximal coordinate neighborhood (U', z) such that $x_i = \frac{\partial}{\partial z_i}$ for $i = 1, \ldots, k$ where $U' \subset U$ and $p \in U'$ and $z(p) = 0$. Hence,

$$D(z^{-1})(e_i) = \frac{\partial}{\partial z_i} = x_i.$$

Now, consider maximal $\epsilon > 0$ such that $[-\epsilon, \epsilon]^k \subset z(U')$. Then, for some constants c_{k+1}, c_n,

$$z^{-1} : [-\epsilon, \epsilon]^k \times \{(c_{k+1}, \ldots, c_n)\} \to M$$

is an integral manifold through the point p. The same process yields integral manifolds through each point q in this integral manifolds; since integral manifolds are unique, taking the union of the images yields an integral submanifold of M. Again, iterating the same process over all points in this union yields another integral manifold, and, by continuing, we get the maximal integral manifold through p. □

4

Let $\mathbb{H}$ be the quaternions and using the fact that $|pq| = |p||q|$ show:

(a): S^3 is a Lie group.

Proof. We know that S^3 is a manifold, so we need to show that S^3 considered as a subset of $\mathbb{H}$ is a group and that the group operation and taking inverses yield smooth maps on S^3. Now, if $p, q \in S^3$, then $|pq| = |p||q| = 1 \cdot 1 = 1$, so $pq \in S^3$ and, thus, S^3 is closed under the group operation. Now, suppose $p = (a, b, c, d)$, $q = (e, f, g, h)$. Then

$$\begin{aligned} pq &= (a, b, c, d) \cdot (e, f, g, h) \\ &= (a + bi + cj + dk)(e + fi + gj + hk) \\ &= (ae - bf - cg - dh) + i(af + be + ch - dg) + j(ag + ce + df - bh) + k(ah + de + bg - cf) \\ &= (ae - bf - cg - dh, af + be + ch - dg, ag + ce + df - bh, ah + de + bg - cf). \end{aligned}$$

The coordinate functions of this map are clearly smooth, so this is a smooth map on $\mathbb{R}^4$, so its restriction to S^3 is also smooth. In general, if $p = a + bi + cj + dk \in \mathbb{H}$, then

$$p^{-1} = \frac{1}{|p|}(a - bi - cj - dk).$$

If $p \in S^3$, then $|p| = 1$, so $p^{-1} = a - bi - cj - dk \in S^3$, thus, the inverse map can be thought of as the map $S^3 \to S^3$ where

$$(a, b, c, d) \mapsto (a, -b, -c, -d);$$

again, this is certainly a smooth map on $\mathbb{R}^4$, so its restriction to S^3 is also smooth. Therefore, we conclude that S^3 is a Lie group. □

(b): $q \to [v \to qvq^{-1}]$ where $q \in \mathbb{H}$, $|q| = 1$ and $v \in Im(\mathbb{H})$ is a homomorphism from S^3 to $SO(3)$ and conclude that $SO(3)$ is diffeomorphic to $\mathbb{RP}^3$.

Proof. First, note that, if $p, q \in S^3$ and f is the map described above, that

$$f(pq) = [v \mapsto (pq)v(pq)^{-1}] = [v \mapsto pqvq^{-1}p^{-1}] = [v \mapsto pvp^{-1}] \circ [v \mapsto qvq^{-1}]$$

so, assuming we can show it is well-defined, f is a group homomorphism. Now, if $q \in S^3$, then $q = (a, b, c, d)$ for some $a, b, c, d \in \mathbb{R}$, which we can re-write as $q = a + bi + cj + dk$. Now, if $v = (1, 0, 0) \in \mathbb{R}^3$, then, considered as an element of $Im(\mathbb{H})$, $v = i$ and

$$\begin{aligned} qvq^{-1} &= (a + bi + cj + dk)i(a - bi - cj - dk) \\ &= (-b + ai + dj - ck)(a - bi - cj - dk) \\ &= i(a^2 + b^2 - c^2 - d^2) + j(2ad + 2bc) + k(2bd - 2ac) \in Im(\mathbb{H}). \end{aligned}$$

Similarly,

$$\begin{aligned} q(j)q^{-1} &= (a + bi + cj + dk)j(a - bi - cj - dk) \\ &= (-c - di + aj + bk)(a - bi - cj - dk) \\ &= i(2bc - 2ad) + j(a^2 + c^2 - b^2 - d^2) + k(2ab + 2cd) \in Im(\mathbb{H}) \end{aligned}$$

and

$$\begin{aligned} q(k)q^{-1} &= (a + bi + cj + dk)k(a - bi - cj - dk) \\ &= (-d + ci - bj + ak)(a - bi - cj - dk) \\ &= i(2ac + 2bd) + j(2cd - 2ab) + k(a^2 + d^2 - b^2 - c^2) \in Im(\mathbb{H}). \end{aligned}$$

Hence, $f(q)$ is given by the matrix

$$(1) \qquad \begin{pmatrix} a^2 + b^2 - c^2 - d^2 & 2bc - 2ad & 2ac + 2bd \\ 2ad + 2bc & a^2 + c^2 - b^2 - d^2 & 2cd - 2ab \\ 2bd - 2ac & 2ab + 2cd & a^2 + d^2 - b^2 - c^2 \end{pmatrix}.$$

Now, we can extend this to a map from $\mathbb{R}^4$ to $M(3, 3, \mathbb{R}) = \mathbb{R}^9$, where $f(a, b, c, d)$ is simply the matrix given above; the coordinate functions of this map are clearly smooth, so f is smooth as a map $\mathbb{R}^4 \to \mathbb{R}^9$; restricting it's domain to S^3 doesn't affect its smoothness. Note that $f(1) = Id_3$ and that, if $q \in S^3$ and since $q^{-1} = a - bi - cj - dk$,

$$f(q^{-1}) = \begin{pmatrix} a^2 + b^2 - c^2 - d^2 & 2bc + 2ad & -2ac + 2bd \\ -2ad + 2bc & a^2 + c^2 - b^2 - d^2 & 2cd + 2ab \\ 2bd + 2ac & -2ab + 2cd & a^2 + d^2 - b^2 - c^2 \end{pmatrix} = f(q)^t.$$

Thus, using the result proved at the beginning of this proof,

$$Id_3 = f(1) = f(qq^{-1}) = f(q)f(q^{-1}) = f(q)f(q)^t,$$

so we see that $f(q) \in O(3)$ for all $q \in S^3$. Furthermore, since $f(1) = Id_3 \in SO(3)$, f is continuous and S^3 is connected, the image of f must be connected and, thus, must lie inside $SO(3)$. Therefore, given that we've shown that f is smooth and that f preserves the group structure, we see that

$$f : S^3 \to SO(3)$$

is a Lie group homomorphism.

Let us now calculate Ker f. If $(a, b, c, d) = q \in$ Ker f, then, setting the matrix in equation (1) equal to the identity, we see that

$$\begin{aligned} a^2 + b^2 - c^2 - d^2 &= 1 \\ a^2 + c^2 - b^2 - d^2 &= 1 \\ a^2 + d^2 - b^2 - c^2 &= 1 \\ a^2 + b^2 + c^2 + d^2 &= 1, \end{aligned}$$

where the last equation comes from the fact that $q \in S^3$. The only solutions to this system of equations are $a = \pm 1$, $b = c = d = 0$, so Ker $f = \{\pm 1\}$ when viewed as a subset of the quaternions. Now, as groups,

$$S^3/(\text{Ker } f) \simeq \text{Image} f;$$

furthermore, since $\{\pm 1\}$ is a discrete subgroup of the center of S^3, we see that $S^3/(\text{Ker } f)$ is in fact a Lie group and since the above isomorphism is given by a restriction of f to this group, this is, in fact, a diffeomorphism.

Now, since Ker $f = \{pm1\}$, $S^3/(\text{Ker } f) = \mathbb{RP}^3$ so the image of f is diffeomorphic to $\mathbb{RP}^3$. Therefore, we need only note that f is surjective to conclude that $SO(3)$ is diffeomorphic to $\mathbb{RP}^3$. □

DRL 3E3A, University of Pennsylvania
E-mail address: shonkwil@math.upenn.edu