GEOMETRY HW 8

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1

Consider the Heisenberg group

$$\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}$$

which is a Lie group diffeomorphic to $\mathbb{R}^3$.

(a): Find the left invariant vector fields X, Y, Z on $\mathbb{R}^3$ whose value at the identity is the standard basis in $\mathbb{R}^3$.

Answer: Note that, if

$$g = \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix},$$

then, if $(x, y, z) \in \mathbb{R}^3$,

$$L_g(x, y, z) = \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & x+a & z+ay+c \\ 0 & 1 & y+b \\ 0 & 0 & 1 \end{pmatrix},$$

which corresponds to $(x + a, y + b, z + ay + c) \in \mathbb{R}^3$. That is,

$$L_g(x, y, z) = (x + a, y + b, a + ay + c),$$

so

$$d(L_g) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & a & 1 \end{pmatrix}.$$

Hence, the left-invariant vector field corresponding to $e_1 = (1, 0, 0)$ in the tangent space of the identity is simply

$$X(g) = d(L_g)_I(e_1) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & a & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = (1, 0, 0) = \frac{\partial}{\partial x}.$$

Similarly,

$$Y(g) = d(L_g)_I(e_2) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & a & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = (0, 1, a) = \frac{\partial}{\partial y} + a\frac{\partial}{\partial z}$$

and

$$Z(g) = d(L_g)_I(e_2) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & a & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = (0,0,1) = \frac{\partial}{\partial z}.$$

♣

(b): Compute the Lie brackets $[X,Y]$, $[X,Z]$ and $[Y,Z]$.

Answer: If H represents the Heisenberg group and $A : (-\epsilon, \epsilon) \to H$ is a path through the origin, then

$$\frac{dA}{dt}|_{t=0} = \begin{pmatrix} 0 & a_1 & a_3 \\ 0 & 0 & a_2 \\ 0 & 0 & 0 \end{pmatrix}$$

for some a_1, a_2, a_3, since the diagonal entries are constant. Hence, elements of the Lie algebra of H, $\mathfrak{h}$, are matrices of this term. Associating with $\mathbb{R}^3$, we see that

$$X(g) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$Y(g) = \begin{pmatrix} 0 & 0 & a \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

and

$$Z(g) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Hence,

$$\begin{aligned} [X,Y] &= XY - YX \\ &= \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & a \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & a \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \end{aligned}$$

Similarly,

$$\begin{aligned} [X,Z] &= XZ - ZX \\ &= \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \end{aligned}$$

and

$$\begin{aligned}[][Y,Z] &= YZ - ZY \\ &= \begin{pmatrix} 0 & 0 & a \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & a \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.\end{aligned}$$

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(c): Find the same Lie brackets in a simpler fashion, using material we learned in class.

Answer: Now, consider $\mathfrak{h}$ as a Lie subalgebra of $\mathfrak{gl}(3,\mathbb{R}) = M(3,3,\mathbb{R}) = \mathbb{R}^9$. Then the standard basis on the copy of $\mathbb{R}^3$, e_1, e_2, e_3, considered above corresponds, when we embed $\mathbb{R}^3$ into $\mathbb{R}^9$ appropriately, to the matrices

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

respectively. In class, we saw that, for $A, B \in M(n,n,\mathbb{R})$, $[A,B] = AB - BA$. Thus, we don't even need to explicitly compute X, Y, Z, as,

$$\begin{aligned}[][X,Y] &= \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\ [X,Z] &= \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\ [Y,Z] &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \\ &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},\end{aligned}$$

which agrees with the results computed in (b) (note that this is valid only because X, Y, Z are left-invariant and thus the Lie brackets are left-invariant as well).

♣

2

(a): Show that the Lie algebra of the Lie group S^3 is isomorphic to the one for $SO(3)$.

Proof. On the last homework, we proved that $\phi : q \mapsto [v \mapsto qvq^{-1}]$ is a surjective Lie group homomorphism from S^3 to $SO(3)$. Thus, ϕ induces a Lie algebra homomorphism $d(\phi) : \mathfrak{s}^3 \to \mathfrak{so}(3)$. Furthermore, we showed that $SO(3)$ is diffeomorphic to $\mathbb{RP}^3$, so S^3 and $SO(3)$ have the same dimension. Therefore, by Sard's theorem, the regular values of ϕ are dense in $SO(3)$. That is to say, $d(\phi)$ is surjective on a dense subset of the tangent spaces. Now, since $d(\phi)$ is linear, this implies that, in fact, $d(\phi)$ is surjective on all of the tangent spaces of $SO(3)$, specifically $d(\phi) : \mathfrak{s}^3 \to \mathfrak{so}(3)$ is surjective. Since $d(\phi)$ is a Lie algebra homomorphism and the dimensions of $\mathfrak{s}^3$ and $\mathfrak{so}(3)$ are equal, this implies that $d(\phi)$ is a vector space isomorphism.

Now, if $V, W \in \mathfrak{s}^3$, then

$$d(\phi)([V,W]) = [d(\phi)(V), d(\phi)(W)]$$

and, if $d(\phi)(V) = Y$, $d(\phi)(W) = Z$, then

$$\begin{aligned} d(\phi)^{-1}([Y,Z]) &= d(\phi)^{-1}([d(\phi)(V), d(\phi)(W)]) \\ &= d(\phi)^{-1}(d(\phi)([V,W])) \\ &= [V,W] \\ &= [d(\phi)^{-1}(Y), d(\phi)^{-1}(Z)], \end{aligned}$$

so $d(\phi)$ preserves the multiplicative structure of the Lie algebras and is, therefore, a Lie algebra isomorphism. □

(b): Show that the Lie algebra of $SO(3)$ and $SU(2)$ are isomorphic, and both are isomorphic to $(\mathbb{R}^3, \times)$, where $\times$ is the cross product.

Proof. Suppose $A = \begin{pmatrix} a+bi & c+di \\ e+fi & g+hi \end{pmatrix} \in SU(2)$. Then

$$\begin{pmatrix} g+hi & -c-di \\ -e-fi & a+bi \end{pmatrix} = A^{-1} = A^* = \begin{pmatrix} a-bi & e-fi \\ c-di & g-hi \end{pmatrix},$$

So $g = a, h = -b, e = -c, d = f$, or

$$\begin{aligned} A &= \begin{pmatrix} a+bi & c+di \\ -c+di & a-bi \end{pmatrix} \\ &= a\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + b\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} + c\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} + d\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}. \end{aligned}$$

Now, if we call these matrices

$$\begin{aligned}\mathbf{1} &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\\ \mathbf{i} &= \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}\\ \mathbf{j} &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\\ \mathbf{k} &= \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}\end{aligned}$$

then it's clear that all elements $\begin{pmatrix} a+bi & c+di \\ -c+di & a-bi \end{pmatrix} \in SU(2)$ can be written in the form

$$a\mathbf{1} + b\mathbf{i} + c\mathbf{j} + d\mathbf{k}.$$

Furthermore, $\mathbf{i}, \mathbf{j}, \mathbf{k}$ are all of order 2 and

$$\begin{aligned}\mathbf{ij} &= \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} = \mathbf{k}\\ \mathbf{ik} &= \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} = -\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = -\mathbf{j}\\ \mathbf{jk} &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} = \mathbf{i},\end{aligned}$$

so we see that the map

$$a\mathbf{1} + b\mathbf{i} + c\mathbf{j} + d\mathbf{k} \mapsto a + bi + cj + dk$$

is a well-defined map from $SU(2)$ to the quaternions. Furthermore,

$$a^2 + b^2 + c^2 + d^2 = \begin{vmatrix} a+bi & c+di \\ -c+di & a-bi \end{vmatrix} = 1$$

so this is a map into the unit quaternions, which correspond to the sphere S^3. This map is clearly bijective and a group homomorphism, so we see that $SU(2)$ and S^3 are isomorphic as groups. Furthermore, both the map and its inverse are smooth since the coordinate functions are smooth, so this is, in fact, a Lie group isomorphism. Therefore, since $SU(2)$ and S^3 are isomorphic as Lie groups, they have the same Lie algebra; we showed in (a) that S^3 and $SO(3)$ have isomorphic Lie algebras, so we conclude that the Lie algebras of $SU(2)$ and $SO(3)$ are isomorphic.

Now, as demonstrated in Spivak (pg. 378), the Lie algebra of $O(3)$ (and, therefore, the connected component containing the identity, namely $SO(3)$) consists of all skew-symmetric matrices. Therefore, define the map $f : \mathfrak{so}(3) \to \mathfrak{r}^3$ by

$$\begin{pmatrix} 0 & -a & -b \\ a & 0 & -c \\ b & c & 0 \end{pmatrix} \mapsto (a, b, c).$$

This map is certainly bijective and, thus, a vector space isomorphism. Furthermore, if $A = \begin{pmatrix} 0 & -a & -b \\ a & 0 & -c \\ b & c & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 & -d & -e \\ d & 0 & -f \\ e & f & 0 \end{pmatrix} \in \mathfrak{so}(3)$, then

$$\begin{aligned} f([A,B]) &= f(AB - BA) \\ &= f\left(\begin{pmatrix} 0 & -a & -b \\ a & 0 & -c \\ b & c & 0 \end{pmatrix}\begin{pmatrix} 0 & -d & -e \\ d & 0 & -f \\ e & f & 0 \end{pmatrix} - \begin{pmatrix} 0 & -d & -e \\ d & 0 & -f \\ e & f & 0 \end{pmatrix}\begin{pmatrix} 0 & -a & -b \\ a & 0 & -c \\ b & c & 0 \end{pmatrix}\right) \\ &= f\left(\begin{pmatrix} -ad-be & -bf & af \\ -ce & -ad-cf & -ae \\ cd & -bd & -be-cf \end{pmatrix} - \begin{pmatrix} -ad-be & -ce & dc \\ -bf & -ad-cf & -bd \\ af & -ae & -be-cf \end{pmatrix}\right) \\ &= f\left(\begin{pmatrix} 0 & ce-bf & af-cd \\ bf-ce & 0 & bd-ae \\ cd-af & ae-bd & 0 \end{pmatrix}\right) \\ &= (bf-ce, dc-af, ae-bd) \\ &= (a,b,c) \times (d,e,f). \end{aligned}$$

Hence, f is a Lie algebra isomorphism. Therefore, we conclude that $\mathfrak{r}^3$ is isomorphic to $\mathfrak{so}(3)$ and, since $\mathfrak{so}(3)$ is isomorphic to $\mathfrak{su}(2)$, to $\mathfrak{su}(2)$ as well. □

3

Show that

(a): $S^1 \times SU(n)$ and $U(n)$ have the same Lie algebra.

Proof. Define the map $f : S^1 \times SU(n) \to U(n)$ by

$$(\lambda, A) \mapsto \lambda A.$$

Since $(\lambda A)^* = \bar{\lambda} A^*$, we see that

$$(\lambda A)(\lambda A)^* = \lambda A \bar{\lambda} A^* = \lambda \bar{\lambda} A A^* = Id,$$

so this map really does map into $U(N)$. Furthermore, if $(\lambda, A), (\gamma, B) \in S^1 \times SU(n)$,

$$f((\lambda, A)(\gamma, B)) = f((\lambda\gamma, AB)) = \lambda\gamma AB = \lambda A \gamma B = f((\lambda, A)) f((\gamma, B)),$$

so this is a group homomorphism. Furthermore, this is clearly a smooth map, so it is a Lie group homomorphism. In addition, if $B \in U(n)$, and we let $x^{1/n}$ denote the principal branch (i.e. $(x^{1/n})^n = x$ and $x^{1/n}$ is the first such value that we encounter when traversing S^1 counter-clockwise starting at 1), then

$$\det\left(\frac{1}{(\det B)^{1/n}} B\right) = \left(\frac{1}{(\det B)^{1/n}}\right)^n \det B = \frac{1}{\det B} \det B = 1$$

and, furthermore,

$$\left(\frac{1}{(\det B)^{1/n}}B\right)\left(\frac{1}{(\det B)^{1/n}}B\right)^t = \frac{1}{(\det B)^{1/n}}\overline{\frac{1}{(\det B)^{1/n}}}BB^t = Id$$

(where $\overline{\frac{1}{(\det B)^{1/n}}}$ is the complex conjugate of $\frac{1}{(\det B)^{1/n}}$); hence,

$$\left(\frac{1}{(\det B)^{1/n}}, \frac{1}{(\det B)^{1/n}}B\right) \mapsto B,$$

so f is surjective. Now, if $(\alpha, A) \in \ker f$, then $\alpha A = Id$, so A is a diagonal matrix. In fact, A must be a scalar matrix, and even at that $(\alpha, A) \in \ker f$ only if α is an nth root of unity ζ_n^i and

$$A = \zeta_n^{n-i} Id.$$

Since $\det(\zeta_n^{n-i} Id) = (\zeta_n^{n-i})^n = 1$, we note that all such pairs are in the kernel of f. Hence,

$$\ker f = \{(\zeta_n^i, \zeta_n^i Id) | \zeta_n \text{ a primitive } n\text{th root of unity}\}$$

so f is n-to-one. Since f is a surjective Lie group homomorphism with discrete kernel, we know that $U(n)$ and $(S^1 \times SU(n))/(\ker f)$ are isomorphic as Lie groups. Furthermore, the argument given in 2(a) above demonstrates that $S^1 \times SU(n)$ and $(S^1 \times SU(n))/(\ker f)$ have the same Lie algebra, so we see that the Lie algebras of $S^1 \times SU(n)$ and $U(n)$ are isomorphic. □

(b): Show that $U(n)$ is diffeomorphic to $S^1 \times SU(n)$ but not isomorphic as Lie groups.

Proof. To see that these two manifolds are not isomorphic as Lie groups, we consider the centers of each. The $Z(U(n))$ consists simply of the scalar matrices. On the other hand, since S^1 is abelian, $Z(S^1 \times SU(n))$ is given by

$$S^1 \times C$$

where C is the set of scalar matrices in $SU(n)$. Now, of the scalar matrices in $U(n)$, the only ones of order n are those with primitive nth roots of unity on their diagonals; since there are $\phi(n)$ such (where ϕ is the Euler phi-function), we see that there are $\phi(n)$ elements of order n in $Z(U(n))$.

On the other hand, $(\zeta_n^i, \zeta_n) \in Z(S^1 \times SU(n))$ for ζ_n a primitive nth root of unity and $1 \leq i \leq n$, and each such element is of order n (since the order of ζ_n^i divides n), so we see that there are at least $n\phi(n)$ elements of order n in $Z(S^1 \times SU(n))$. Hence, we conclude that $Z(U(n))$ is not isomorphic to $Z(S^1 \times SU(n))$ and, therefore, that $U(n)$ and $S^1 \times SU(n)$ are not isomorphic as groups and, hence, they are not isomorphic as Lie groups.

Now, to show that these two Lie groups are diffeomorphic, define the map $f : U(n) \to S^1 \times SU(n)$ by

$$A \mapsto \left(e^{2\pi i\alpha}, \begin{pmatrix} e^{-2\pi i\alpha} & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix} A \right),$$

where $e^{2\pi i\alpha} = \det A$. Note that

$$\begin{aligned} \det\left(\begin{pmatrix} e^{-2\pi i\alpha} & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix} A \right) &= \det\left(\begin{pmatrix} e^{-2\pi i\alpha} & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix} \right) \det A \\ &= e^{-2\pi i\alpha} e^{2\pi i\alpha} \\ &= 1. \end{aligned}$$

Also, since taking the conjugate transpose reverses the order, it is clear that the image of A under ϕ is self-adjoint, so we see that this really is a map into $S^1 \times SU(n)$. Since the determinant map is continuous, this is a smooth map. Now, if we define $\psi : S^1 \times SU(n) \to U(n)$ by

$$(e^{2\pi i\alpha}, A) \mapsto \begin{pmatrix} e^{2\pi i\alpha} & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix} A$$

then ψ is also clearly a smooth map. Furthermore,

$$\begin{aligned} \psi \circ \phi(A) &= \psi\left(e^{2\pi i\alpha}, \begin{pmatrix} e^{-2\pi i\alpha} & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix} A \right) \\ &= \begin{pmatrix} e^{2\pi i\alpha} & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix} \begin{pmatrix} e^{-2\pi i\alpha} & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix} A \\ &= IdA \\ &= A \end{aligned}$$

and

$$\begin{aligned}\phi\circ\psi(e^{2\pi i\alpha},A) &= \phi\left(\begin{pmatrix} e^{2\pi i\alpha} & 0 & \cdots & 0\\ 0 & 1 & \cdots & 0\\ \vdots & & \ddots & \vdots\\ 0 & 0 & \cdots & 1\end{pmatrix} A\right)\\ &= \left(e^{2\pi i\alpha}, \begin{pmatrix} e^{-2\pi i\alpha} & 0 & \cdots & 0\\ 0 & 1 & \cdots & 0\\ \vdots & & \ddots & \vdots\\ 0 & 0 & \cdots & 1\end{pmatrix}\begin{pmatrix} e^{2\pi i\alpha} & 0 & \cdots & 0\\ 0 & 1 & \cdots & 0\\ \vdots & & \ddots & \vdots\\ 0 & 0 & \cdots & 1\end{pmatrix} A\right)\\ &= (e^{2\pi i\alpha},A)\end{aligned}$$

so $\phi\circ\psi = Id$ and $\psi\circ\phi = Id$; since both are smooth, we see that $\psi = \phi^{-1}$ and that ϕ is a diffeomorphism, so $S^1\times SU(n)$ is diffeomorphic to $U(n)$. □

4

Show that every Lie group is orientable.

Proof. Let G be a Lie group and let $\{e_1,\dots,e_n\}$ denote a basis for T_eG. Then let $V_1,\dots,V_n$ denote the left invariant vector fields on G such that

$$V_i(e) = e_i$$

for all $i = 1,\dots,n$. Then, since the X_i are left-invariant and L_{g*} gives a vector space isomorphism on the tangent spaces, we see that the X_i must be linearly independent everywhere. Hence, for all $(a,v)\in TG$, $(a,v) = \sum_{i=1}^n a_iX_i(a)$, so we can define the map $f : TG\to G\times\mathbb{R}^n$, where

$$\sum_{i=1}^n a_iX_i(a)\mapsto(a,a_1,\dots,a_n).$$

The second coordinate is clearly linear and this map is certainly smooth with smooth inverse, so we see that TG is trivial and, thus, that G is necessarily orientable. □

5

Show that $F : S^3\to S^3\to SO(4)$ defined by $F(q,r) = \{v\in\mathbb{H}\to qvr^{-1}\in\mathbb{H}\}$ is a homomorphism which is onto with kernel $\{\pm(1,1)\}$. Conclude that $SO(4)$ is diffeomorphic to $S^3\times S^3/\{(x,y)\sim-(x,y)\}$.

Proof. First, note that, if $(q,r),(p,s)\in S^3\times S^3$, then

$$\begin{aligned}F((q,r)(p,s)) = F(qp,rs) &= [v\mapsto qpv(rs)^{-1}]\\ &= [v\mapsto qpvs^{-1}r^{-1}]\\ &= [v\mapsto qvr^{-1}]\circ[v\mapsto pvs^{-1}]\\ &= F(q,r)\circ F(p,s),\end{aligned}$$

so, assuming we can show it is well-define, F is a group homomorphism. Now, if $q = a + bi + cj + dk$ and $r = e + fi + gj + hk$, then $r^{-1} = \bar{r}$, so:

$$\begin{aligned}
q(1)r^{-1} &= (a + bi + cj + dk)(e - fi - gj - hk) \\
&= (ae + bf + cg + dh) + i(be - af - ch + dg) \\
&\quad +j(ce - ag - df + bh) + k(de - ah - bg + cf) \\
q(i)r^{-1} &= (a + bi + cj + dk)i(e - fi - gj - hk) \\
&= (a + bi + cj + dk)(f + ei + hj - gk) \\
&= (af - be - ch + dg) + i(ae + bf - cg - dh) \\
&\quad +j(ah + cf + de + bg) + k(df - ag + bh - ce) \\
q(j)r^{-1} &= (a + bi + cj + dk)j(e - fi - gj - hk) \\
&= (a + bi + cj + dk)(g - hi + ej + fk) \\
&= (ag + bh - ce - df) + i(bf - ah + cf - de) \\
&\quad +j(ae + cg - dh - bf) + k(af + dg + be + ch) \\
q(k)r^{-1} &= (a + bi + cj + dk)k(e - fi - gj - hk) \\
&= (a + bi + cj + dk)(h + gi - fj + ek) \\
&= (ah - bg + cf - de) + i(ag + bh + ce + df) \\
&\quad +j(ch - af - be + dg) + k(ae + dh - bf - cg).
\end{aligned}$$

Hence, $F(q, r)$ is given by the matrix

$$\begin{pmatrix} ae + bf + cg + dh & af - be - ch + dg & ag + bh - ce - df & ah - bg + cf - de \\ be - af - ch + dg & ae + bf - cg - dh & bf - ah + cf - de & ag + bh + ce + df \\ ce - ag - df + bh & ah + cf + de + bg & ae + cg - dh - bf & ch - af - be + dg \\ de - ah - bg + cf & df - ag + bh - ce & af + dg + be + ch & ae + dh - bf - cg \end{pmatrix}.$$

Now, we can extend this to a map from $S^3 \times S^3$ to $\mathbb{R}^{16}$; the coordinate functions of this map are clearly smooth, so restricting the range doesn't affect smoothness, and, hence, we see that F is smooth.

If (q, r) as above, then $(q, r)^{-1} = (q^{-1}, r^{-1}) = (\bar{q}, \bar{r})$, so, substituting appropriately in the above matrix, we see that $F((q, r))^{-1} = F((q, r))^t$, so the image of F really is contained in $O(4)$. Furthermore, $F(1, 1) = Id_4$ and thus, since F is continuous and $S^3 \times S^3$ is connected, the image of F must be contained in the connected component of $O(4)$ containing the identity, namely $SO(4)$. Therefore, $F : S^3 \times S^3 \to SO(4)$ is a Lie group homomorphism.

Now, if (q, r) as above (in terms of $a, b, \ldots$) and $(q, r) \in \ker F$, then, looking at the diagonal entries and recalling that $q\bar{q} = r\bar{r} = 1$, we see that

$$\begin{aligned}
ae + bf + cg + dh &= 1 \\
ae + bf - cg - dh &= 1 \\
ae + cg - dh - bf &= 1 \\
ae + dh - bf - cg &= 1 \\
a^2 + b^2 + c^2 + d^2 &= 1 \\
e^2 + f^2 + g^2 + h^2 &= 1.
\end{aligned}$$

Thus, it must be the case that $ae = 1$ and $b = c = d = f = g = h = 0$, so we see that

$$(q, r) = \pm(1, 1);$$

that is, $\ker F = \{\pm(1,1)\}$. Therefore, if only we can show that F is surjective, we will be able to conclude that

$$SO(4) \approx (S^3 \times S^3)/\ker F = (S^3 \times S^3)/\{(x,y) \sim -(x,y)\}.$$

To do so, it suffices to show that, for an arbitrary oriented, orthonormal basis of $\mathbb{R}^4$ there exists $[v \mapsto qvr^{-1}] \in \text{Image } F$ such that $[v \mapsto qvr^{-1}]$ maps the arbitrary basis to the standard basis, $\{1,i,j,k\}$. To that end, let $\{w,x,y,z\}$ be an oriented, orthonormal basis for $\mathbb{R}^4$. Let $q = rw^{-1}$. Then

$$F(q,r)(w,x,y,z) = (1, rw^{-1}xr^{-1}, rw^{-1}yr^{-1}, rw^{-1}zr^{-1}). \tag{1}$$

Now, since w is of unit length and linearly independent of x, y, z, $\{1, w^{-1}x, w^{-1}y, w^{-1}z\}$ is also an oriented basis for $\mathbb{R}^4$. Thus, we need only find r such that the right side of equation (1) above is simply $(1,i,j,k)$. However, this is precisely what we needed to show on the last homework to see that our map from S^3 to $SO(3)$ was surjective; having reduced to this problem, then, we see that, in fact, F is surjective. Therefore, we conclude that, indeed, $SO(4)$ is diffeomorphic to $(S^3 \times S^3)/\{(x,y) \sim -(x,y)\}$. $\square$

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