GEOMETRY HW 9

CLAY SHONKWILER

1

Show that the tensor product of 2 vector spaces is well-defined by the universality property for bilinear forms.

Proof. Let V and W be vector spaces and let $\pi : V \times W \to V \otimes W$. Suppose Z is a vector space that satisfies the universality property for bilinear forms; that is, there is a surjective linear map $\pi' : V \times W \to Z$ and for all vector spaces X, if $f : V \times W \to X$ is a bilinear map, then $f = g \circ \pi'$ for a unique linear map $g : Z \to X$. Now, note that $\pi : V \times W \to V \otimes W$ is bilinear: if $v_1, v_2 \in V$, $w_1, w_2 \in W$, $a, b \in \mathbb{R}$, then

$$\pi(av_1+bv_2, w_1) = (av_1+bv_2)\otimes w_1 = a(v_1\otimes w_1)+b(v_2\otimes w_1) = a\pi(v_1, w_1)+b\pi(v_2, w_1)$$

and

$$\pi(v_1, aw_1+bw_2) = v_1\otimes(aw_1+bw_2) = a(v_1\otimes w_1)+b(v_1\otimes w_2) = a\pi(v_1, w_1)+b\pi(v_1, w_2).$$

Hence, there exists a unique linear map $g' : Z \to V\otimes W$ such that $\pi = g' \circ \pi'$. Since π is surjective, g' must also be surjective. Thus, since we're dealing with finite-dimensional vector spaces and g' is linear, g' is bijective. Since g' is a bijective linear map, it is a vector space isomorphism, so we see that $Z \simeq V \otimes W$ as vector spaces, so the tensor product is well-defined by the universal property up to isomorphism. □

2

If V is an n-dimensional vector space and $A : V \to V$ a linear map,

(a): Show that the induced map on $\Lambda_k(V)$ is well-defined by

$$v_1 \wedge \ldots \wedge v_k \mapsto A(v_1) \wedge \ldots \wedge A(v_k).$$

Proof. Define the map $F : V \times \ldots \times V \to \Lambda_k(V)$ by

$$(v_1, \ldots, v_k) \mapsto A(v_1) \wedge \ldots \wedge A(v_k).$$

Now, suppose $v_1, \ldots, v_k \in V$ such that $v_i = v_{i+1}$ for some i. Then $A(v_i) = A(v_{i+1})$, so

$$F(v_1, \ldots, v_k) = A(v_1) \wedge \ldots \wedge A(v_i) \wedge A(v_i) \wedge \ldots \wedge A(v_k) = 0,$$

so F is an alternating map. Now suppose $v_1, \dots, v_k, v_i' \in V$ and $a, b \in \mathbb{R}$. Then

$$\begin{aligned} F(v_1, \dots, av_i + bv_i', \dots, v_k) &= A(v_1) \wedge \dots \wedge A(av_i + bv_i') \wedge \dots \wedge A(v_k) \\ &= A(v_1) \wedge \dots \wedge aA(v_i) + bA(v_i') \wedge \dots \wedge A(v_k) \\ &= A(v_1) \wedge \dots \wedge aA(v_i) \wedge \dots \wedge A(v_k) \\ &\quad + A(v_1) \wedge \dots \wedge bA(v_i') \wedge \dots \wedge A(v_k) \\ &= a(A(v_1) \wedge \dots \wedge A(v_i) \wedge \dots \wedge A(v_k)) \\ &\quad + b(A(v_1) \wedge \dots \wedge A(v_i') \wedge \dots \wedge A(v_k)) \\ &= aF(v_1, \dots, v_i, \dots, v_k) + bF(v_1, \dots, v_i', \dots, v_k) \end{aligned}$$

since A is linear, so F is multilinear. Therefore, since F is an alternating, multilinear map, the universal property of the exterior algebra tells us that $F = G \circ \pi$ for a unique linear map $G : \Lambda_k(V) \to \Lambda_k(V)$. That is

$$A(v_1) \wedge \dots \wedge A(v_k) = F(v_1, \dots, v_k) = G \circ \pi(v_1, \dots, v_k) = G(v_1 \wedge \dots \wedge v_k),$$

so we see that G is just $\Lambda_k(A)$. Therefore, $\Lambda_k(A)$ is well-defined. □

(b): Show that the map on $\Lambda_n(V)$ is multiplication by $\det(A)$.

Proof. We know that $\Lambda_n(V) \simeq \mathbb{R}$, because, if $v_1, \dots, v_n$ is a basis for V, thn $v_1 \wedge \dots \wedge v_n$ is a basis for $\Lambda_n(V)$ (see problem 3 below). Hence, all elements of $\Lambda_n(V)$ are of the form $a(v_1 \wedge \dots \wedge v_n)$ for some $a \in \mathbb{R}$. Hence, if $A : V \to V$ is linear, $u_1, \dots, u_n \in V$,

$$\Lambda_n(A)(u_1 \wedge \dots \wedge u_n) = D(A)(v_1 \wedge \dots \wedge v_n)$$

where $D(A) \in \mathbb{R}$. Now, V is isomorphic, as a vector space, to $\mathbb{R}^n$, so, since A is a linear map, we can think of A as an $n \times n$ matrix over $\mathbb{R}$. If we consider the columns of the matrix of A as vectors in $\mathbb{R}^n$, D is a map from $M_{n,n}(\mathbb{R}) = \mathbb{R}^n \times \dots \times \mathbb{R}_n$ to $\mathbb{R}$.

Note that, since all elements of $\Lambda_n(V)$ are simply multiples of $v_1 \wedge \dots \wedge v_n$, it suffices to show that

$$\Lambda_n(A)(v_1 \wedge \dots \wedge v_n) = (\det A) v_1 \wedge \dots \wedge v_n.$$

Now, if A is the identity map I, then

$$\Lambda_n(A)(v_1 \wedge \dots \wedge v_n) = A(v_1) \wedge \dots \wedge A(v_n) = v_1 \wedge \dots \wedge v_n,$$

so $D(I) = 1$.

Suppose A is such that the matrix of A has the ith and $(i+1)$st columns equal. Then $A(v_i) = A(v_{i+1})$, so

$$\Lambda_n(A)(v_1 \wedge \dots \wedge v_n) = A(v_1) \wedge \dots \wedge A(v_n) = A(v_1) \wedge \dots \wedge A(v_i) \wedge A(v_i) \wedge \dots \wedge A(v_n) = 0,$$

so $D(A) = 0$. Hence, D is an alternating map.

Finally, suppose the columns of A are given by w_i for $i = 1, \ldots, n$, where $w_i = u + v$. Then

$$\begin{aligned} D(A)(v_1 \wedge \ldots \wedge v_n) &= \Lambda_n(A)(v_1 \wedge \ldots \wedge v_n) \\ &= A(v_1) \wedge \ldots \wedge A(v_n) \\ &= w_1 \wedge \ldots \wedge w_n \\ &= w_1 \wedge \ldots \wedge (u+v) \wedge \ldots w_n \\ &= w_1 \wedge \ldots \wedge u \wedge \ldots \wedge w_n + w_1 \wedge \ldots \wedge v \wedge \ldots \wedge w_n \\ &= D([w_1 \ldots u \ldots w_n])(v_1 \wedge \ldots \wedge v_n) + D([w_1 \ldots v \ldots w_n])(v_1 \wedge \ldots \wedge v_n) \\ &= (D([w_1 \ldots u \ldots w_n]) + D([w_1 \ldots v \ldots w_n]))(v_1 \wedge \ldots \wedge v_n), \end{aligned}$$

so

$$D([w_1 \ldots u \ldots w_n]) + D([w_1 \ldots v \ldots w_n]) = D(A) = D([w_1 \ldots u+v \ldots w_n]);$$

hence, D is multilinear. Now, since the determinant map is the unique alternating, multilinear map from $M_{n,n}(\mathbb{R})$ to $\mathbb{R}$ taking the identity to 1, we see that D must be the determinant map; i.e. $D(A) = \det A$. Therefore, $\Lambda_n(A)$ is simply multiplication by $\det A$. □

3

Show that the basis for $\Lambda_k(V)$ induced by a basis of V is indeed a basis.

Proof. Let $v_1, \ldots, v_n$ be a basis for V. Consider a simple term

$$u_1 \wedge \ldots u_k \in \Lambda_k(V).$$

Then, for each u_i,

$$u_i = \sum_{j=1}^{n} a_j^i v_j.$$

Hence,

$$\begin{aligned} u_1 \wedge \ldots \wedge u_n &= (\textstyle\sum_1^n a_j^1 v_j) \wedge \ldots \wedge (\sum_1^n a_j^n v_j) \\ &= \textstyle\sum_{k_1=1}^n \left(a_{k_1}^1 v_{k_1} \wedge \left(\sum_1^n a_j^2 v_j\right) \wedge \ldots \wedge \left(\sum_1^n a_j^n v_j\right)\right) \\ &= \textstyle\sum_{k_1=1}^n a_{k_1}^1 \left(v_{k_1} \wedge \left(\sum_1^n a_j^2 v_j\right) \wedge \ldots \wedge \left(\sum_1^n a_j^n v_j\right)\right) \\ &= \textstyle\sum_{k_1=1}^n \sum_{k_2=1}^n a_{k_1}^1 a_{k_2}^2 \left(v_{k_1} \wedge v_{k_2} \wedge \left(\sum_1^n a_j^3 v_j\right) \wedge \ldots \wedge \left(\sum_1^n a_j^n v_j\right)\right) \\ &\vdots \\ &= \textstyle\sum_{k_1=1}^n \cdots \sum_{k_n=1}^n a_{k_1}^1 \cdots a_{k_n}^n \left(v_1 \wedge \ldots \wedge v_n\right). \end{aligned}$$

Hence, all simple elements of the form $u_1 \wedge \ldots \wedge u_k$ can be written as a linear combination of terms of the form

$$v_{i_1} \wedge \ldots \wedge v_{i_k}.$$

Furthermore, since we can interchange across the $\wedge$'s by introducing a sign change, this means each simple term of the form $u_1 \wedge \ldots \wedge u_k$ can be written as a linear combination of terms of the form

$$v_{i_1} \wedge \ldots \wedge v_{i_k}$$

where $1 \leq i_1 \leq i_2 \leq \ldots \leq i_k \leq n$. However, if $i_j = i_{j+1}$ (i.e. if any of the middle inequalities aren't strict), then $v_{i_1} \wedge \ldots v_{i_k} = 0$, so, in fact, we can make the further restriction that $1 \leq i_1 < i_2 < \ldots < i_k \leq n$. Now, since any simple tensor can be written in this form, it's clear that a linear combination of simple tensors will simply be another term of this form, so we see that

$$\{v_{i_1} \wedge \ldots \wedge v_{i_k} | 1 \leq i_1 < \ldots < i_k \leq n\}$$

spans $\Lambda_k(V)$.

To show that these generators are linearly independent, it suffices to exhibit a linear map $F : \Lambda_k(V) \to \mathbb{R}$ that takes the value 1 at a given generator $v_{i_1} \wedge \ldots \wedge v_{i_k}$ and zero on all other generators in the above set. To that end, define $f : V \times \ldots \times V \to \mathbb{R}$ by

$$(v_{j_1}, \ldots, v_{j_k}) \mapsto \text{sgn}\sigma$$

if σ is the unique permutation sending the ordered k-tuple $(j_1, \ldots, j_k)$ to $(i_1, \ldots, i_k)$ (sgnσ is the sign of σ) and

$$f(v_{j_1}, \ldots, v_{j_k}) = 0$$

if the tuple $(j_1, \ldots, j_k)$ cannot be permuted into $(i_1, \ldots, i_k)$; extend f multilinearly to all of $V \times \ldots \times V$. Note that

$$f(u_1, \ldots, u_k) = f(u_{\sigma(1)}, \ldots, u_{\sigma(k)}),$$

since we used sgnσ, so this map is, in fact, alternating. Therefore, since alternating multilinear maps factor through $\Lambda_k(V)$, we see that there exists linear $F : \Lambda_k(V) \to \mathbb{R}$ such that $f = F \circ \pi$, where $\pi : V \times \ldots \times V \to \Lambda_k(V)$ i the usual projection. However,

$$F(v_{i_1} \wedge \ldots \wedge v_{i_k}) = f(v_{i_1}, \ldots, v_{i_k}) = 1$$

and for any other generating element $v_{j_1} \wedge \ldots \wedge v_{j_k}$,

$$F(v_{j_1} \wedge \ldots \wedge v_{j_k}) = f(v_{j_1}, \ldots, v_{j_k}) = 0,$$

since $(j_1, \ldots, j_k)$ cannot be permuted into $(i_1, \ldots, i_k)$, due to the fact that the i's and j's are strictly increasing and, therefore, the j's cannot all come from the set $\{i_1, \ldots, i_k\}$. Thus, F is exactly the desired map, so we see that all elements of the generating set given above are linearly independent, and so

$$\{v_{i_1} \wedge \ldots \wedge v_{i_k} | 1 \leq i_1 < \ldots < i_k \leq n\}$$

is a basis for $\Lambda_k(V)$.

□

4

Given a vector bundle E over M. Define a vector bundle $\Lambda_k(E)$ and prove it is a vector bundle.

Proof. Let $\pi : E \to M$ be a vector bundle with trivializations given by

$$f_p : \pi^{-1}(U) \to U \times \mathbb{R}^n.$$

Define

$$\Lambda_k(E) = \bigcup_{p \in M} \Lambda_k(E_p)$$

where $E_p = \pi^{-1}(p) \subset E$. Now, define the map $\sigma : \Lambda_k(E) \to M$, where $\sigma(\Lambda_k(E_p)) = p$. Now, since f_p is a trivialization, the restriction

$$f_p|_p : E_p \to \{p\} \times \mathbb{R}^n$$

is a vector space isomorphism. Now, our work in problem 2 above demonstrates that the induced linear map $\Lambda_k(f_p|_p) : \Lambda_k(E_p) \to \{p\} \times \Lambda_k(\mathbb{R}^n)$ is well-defined by

$$v_1 \wedge \ldots \wedge v_k \mapsto f_p(v_1) \wedge \ldots \wedge f_p(v_k).$$

Now, suppose $(p, r_1 \wedge \ldots \wedge r_k) \in \{p\} \times \Lambda_k(\mathbb{R}^n)$. Then, since $f_p|_p$ is a vector space isomorphism, there exist $v_i \in E$ such that $f_p(v_i) = (p, w_i)$ for each $i = 1, \ldots, k$. Thus,

$$\Lambda_k(f_p|_p)(v_1 \wedge \ldots \wedge v_k) = (p, r_1 \wedge \ldots \wedge r_k),$$

so we see that $\Lambda_k(f_p|_p)$ is surjective. Since it is a surjective, linear map, $\Lambda_k(f_p)$ is, in fact, a vector space isomorphism and, therefore, has a linear inverse (strictly speaking, we should say "linear in the second coordinate"). Note that $\Lambda_k(f_p|_p)^{-1} = \Lambda_k(f_p^{-1}|_p)$.

Now, since $\Lambda_k(\mathbb{R}^n)$ is isomorphic to $\mathbb{R}^{\binom{n}{k}}$, we will fix a particular isomorphism $\phi : \Lambda_k(\mathbb{R}^n) \to \mathbb{R}^{\binom{n}{k}}$. Then, for all $p \in M$,

$$\Lambda_k(f_p|_p) : \Lambda_k(E_p) \to \{p\} \times \mathbb{R}^{\binom{n}{k}}$$

is a vector space isomorphism. Therefore, if U is the neighborhood of p such that $f_p : \pi^{-1}(U) \to U \times \mathbb{R}^n$ is a trivialization, then we can extend $\Lambda_k(f_p|_p)$ to $\Lambda_k(f_p) : \sigma^{-1}(U) \to U \times \mathbb{R}^{\binom{n}{k}}$; note that for a fixed base point $q \in U$, $\Lambda_k(f_p)$ on $\sigma^{-1}(q)$ is just given by $\Lambda_k(f_q|q)$, so we see that $\Lambda_k(f_p)$ is linear in the second coordinate. Since in the first coordinate it's just the identity map, we see that

$$\Lambda_k(f_p) : \sigma^{-1}(U) \to U \times \mathbb{R}^{\binom{n}{k}}$$

is clearly differentiable.

Now, suppose U is a neighborhood of p and V is a neighorhood of q such that

$$f_p : \pi^{-1}(U) \to U \times \mathbb{R}^n$$

and

$$f_q : \pi^{-1}(V) \to V \times \mathbb{R}^n$$

are trivializations and $U \cap V \neq \emptyset$. Then

$$f_q \circ f_p^{-1} : (U \cap V) \times \mathbb{R}^n \to (U \cap V) \times \mathbb{R}^n$$

is linear. Then we know that

$$\Lambda_k(f_q) \circ \Lambda_k(f_p)^{-1} : (U \cap V) \times \mathbb{R}^{\binom{n}{k}} \to (U \cap V) \times \mathbb{R}^{\binom{n}{k}}$$

is well-defined. Furthermore, for $(x, v_1 \wedge \ldots \wedge v_{\binom{n}{k}}) \in (U \cap V) \times \mathbb{R}^{\binom{n}{k}}$, we know that

$$\begin{aligned}
\Lambda_k(f_q) \circ \Lambda_k(f_p)^{-1}(x, v_1 \wedge \ldots \wedge v_{\binom{n}{k}}) &= \Lambda_k(f_q) \circ \Lambda_k(f_p^{-1})(x, v_1 \wedge \ldots \wedge v_{\binom{n}{k}}) \\
&= \Lambda_k(f_q)(f_p^{-1}(v_1) \wedge \ldots \wedge f_p^{-1}(v_{\binom{n}{k}})) \\
&= (x, f_q \circ f_p^{-1}(v_1) \wedge \ldots \wedge f_q \circ f_p^{-1}(v_{\binom{n}{k}})) \\
&= \Lambda_k(f_q \circ f_p^{-1})(x, v_1 \wedge \ldots \wedge v_{\binom{n}{k}}),
\end{aligned}$$

which is to say $\Lambda_k(f_q) \circ \Lambda_k(f_p)^{-1} = \Lambda_k(f_q \circ f_p^{-1})$, which is linear since $f_q \circ f_p^{-1}$ is.

Therefore, we see that the $\Lambda_k(f_p) : \sigma^{-1}(U) \to U \times \mathbb{R}^{\binom{n}{k}}$ give trivializations, so $\sigma : \Lambda_k(E) \to M$ is a vector bundle. □

5

Show that for an n-dimensional vector bundle E, a non-zero section of $\Lambda_n(E)$ (non-zero at every point) defined an orientation on E.

Proof. Recall that $\Lambda_n(E) = \cup_{p \in M} \Lambda_n(E_p)$ and that $\Lambda_n(E_p) = \{r(v_1 \wedge \ldots \wedge v_n)\}$ where $r \in \mathbb{R}$ and $v_1, \ldots, v_n$ is a basis for E_p. Hence, a section $\tau : E \to \Lambda_n(E)$ is simply given by

$$\tau(v) = a(v)(v_1 \wedge \ldots \wedge v_n)$$

where $a : E \to \mathbb{R}$ is smooth. Since τ is non-zero at every point, we see that $a(v) \neq 0$ for all $v \in E$. By the Intermediate Value Theorem, then, a is strictly positive or strictly negative. Let

$$[v_1, \ldots, v_n]$$

define an orientation on E_p for all p.

Now, if $t : \pi^{-1}(U) \to U \times \mathbb{R}^n$ is a trivialization, then $t|_p : E_p \to \{p\} \times \mathbb{R}^n$ is a vector space isomorphism; thus, the induced map (restricted to $\sigma^{-1}(p)$ for the moment)

$$\Lambda_n(t) : \sigma^{-1}(p) \to \{p\} \times \Lambda_n(\mathbb{R}^n),$$

is a vector space isomorphism as in problem 4 above. By our work in 2(b) above, we know that $\Lambda_n(t)$ is simply given by multiplication by the determinant of t. That is,

$$\Lambda_n(t)(v_1 \wedge \ldots \wedge v_n) = \det(t) e_1 \wedge \ldots \wedge e_n.$$

Now,

$$\Lambda_n(t) \circ \tau \circ t^{-1} : \{p\} \times \mathbb{R}^n \to \{p\} \times \Lambda_n(\mathbb{R}^n)$$

is a well-defined smooth map, since t with this restriction is a vector space isomorphism. Furthermore, since τ is non-zero at every point in E and $\Lambda_n(t)$

is an injective linear map, $F = \Lambda_n(t) \circ \tau \circ t^{-1}$ is non-zero at every point. Now, since $\Lambda_n(\mathbb{R}^n)$ is generated by $\{e_1 \wedge \ldots \wedge e_n\}$,

$$F(p, u) = (p, b(u)e_1 \wedge \ldots \wedge e_n)$$

for $b : \mathbb{R}^n \to \mathbb{R}$. Since F is non-zero, b must be strictly positive or strictly negative at every point. Therefore, for $v \in E_q$, where $q \in U$,

$$\begin{aligned}(\det(t)a(v))e_1 \wedge \ldots \wedge e_n \quad &= \Lambda_n(t)(a(v)v_1 \wedge \ldots \wedge v_n) \\ &= (\Lambda_n(t) \circ \tau)(v) \\ &= (\Lambda_n(t) \circ \tau \circ t^{-1} \circ t)(v) \\ &= (F \circ t)(v) \\ &= F(q, t(v)) \\ &= b(t(v))e^1 \wedge \ldots \wedge e_n,\end{aligned}$$

so

$$\det(t)a(v) = b(t(v)).$$

Since a and b are both either strictly positive or strictly negative, we see that $\det(t)$ is either strictly positive or strictly negative and, hence, is either orientation preserving or orientation reversing.

Our choice of U was arbitrary, so we see the same holds true for all such trivializations $t : \pi^{-1}(U) \to U \times \mathbb{R}^n$, which is precisely what it means to define an orientation on E. □

DRL 3E3A, University of Pennsylvania
E-mail address: shonkwil@math.upenn.edu