GEOMETRY FINAL

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1

(a): If M is non-orientable and $p \in M$, is $M - \{p\}$ orientable?

Answer: No. Suppose $M - \{p\}$ is orientable, and let (U_α, x_α) be an atlas that gives an orientation on $M - \{p\}$. Now, let (V, y) be a coordinate chart on M (in some atlas) containing the point p. If (U, x) is a coordinate chart such that $U \cap V \neq \emptyset$, then, since x and y are both smooth maps with smooth inverses, the composition

$$x \circ y^{-1} : y(U \cap V) \to x(U \cap V)$$

is a smooth map. Hence, the collection $\{(U_\alpha, x_\alpha), (V, \alpha)\}$ gives an atlas on M. Now, since $M - \{p\}$ is orientable,

$$\det \left(\frac{\partial x_\alpha^i}{\partial x_\beta^j} \right) > 0$$

for all α, β. However, since M is non-orientable, there must exist coordinate charts where the Jacobian is negative on the overlap; clearly, one of each such pair must be (V, y). Hence, there exists β such that

$$\det \left(\frac{\partial y^i}{\partial x_\beta^j} \right) \leq 0.$$

Suppose this is true for all β such that $U_\beta \cap V \neq \emptyset$. Then, since $y(q) = (y^1(q), \ldots, y^n(q))$, let $\tilde{y}$ be such that $\tilde{y}(p) = (y^2(p), y^1(p), y^3(p), y^4(p), \ldots, y^n(p))$. Then $\tilde{y} : V \to \tilde{y}(V)$ is a diffeomorphism, so $(V, \tilde{y})$ gives a coordinate chart containing p and

$$\det \left(\frac{\partial \tilde{y}^i}{\partial x_\beta^j} \right) = -\det \left(\frac{\partial y^i}{\partial x_\beta^j} \right) \geq 0$$

for all β such that $U_\beta \cap V \neq \emptyset$. Hence, unless equality holds in some case, $\{(U_\alpha, x_\alpha), (V, \tilde{y})\}$ defines an orientation on M, contradicting the fact that M is non-orientable.

Additionally, it's clear that equality can never hold, since $y : U_\beta \cap V \to y(U_\beta \cap V)$ is a diffeomorphism and, thus, has full rank.

Therefore, the above contradiction implies that there exist β, γ such that

$$\det\left(\frac{\partial y^i}{\partial x^j_\beta}\right) > 0 \qquad \det\left(\frac{\partial y^i}{\partial x^j_\gamma}\right) < 0.$$

Let c be a path in V from $q_1 \in U_\beta \cap V$ to $q_2 \in U_\gamma \cap V$ such that c does not pass through p. Then the U_α cover c and c is compact, so a finite subcollection $U_1, \dots, U_k$ covers c, where $U_1 = U_\beta$ and $U_k = U_\gamma$. Since $\det\left(\frac{\partial y^i}{\partial x^j_1}\right) > 0$ and $\det\left(\frac{\partial y^i}{\partial x^j_k}\right) < 0$, there must exist $l \in \{1, \dots, k\}$ such that

$$\det\left(\frac{\partial y^i}{\partial x^j_l}\right) > 0 \qquad \det\left(\frac{\partial y^i}{\partial x^j_{l+1}}\right) < 0.$$

Now, $W = U_l \cap U_{l+1} \cap V \neq \emptyset$, since U_l and U_{l+1} overlap on some segment of c, which is contained in V. Note that

$$\left(\frac{\partial y^i}{\partial x^j_{l+1}}\right) = \left(\frac{\partial y^i}{\partial x^j_l}\right)\left(\frac{\partial x^i_l}{\partial x^j_{l+1}}\right)$$

by the chain rule, so

$$0 > \det\left(\frac{\partial y^i}{\partial x^j_{l+1}}\right) = \det\left(\frac{\partial y^i}{\partial x^j_l}\right) \cdot \det\left(\frac{\partial x^i_l}{\partial x^j_{l+1}}\right) > 0.$$

From this contradiction, then, we conclude that $M - \{p\}$ is not orientable.

♣

(b): Let $f : M \to N$ be a local diffeomorphism. If one of M or N is orientable, is the other orientable?

Answer: If N is orientable, then so is M. If M is orientable, then N need not be. Suppose $f : M \to N$ is a local diffeomorphism and that N is orientable. Then, at each point $p \in M$, there exists a neighborhood U_p of p such that $f : U_p \to f(U_p)$ is a diffeomorphism. Let (V_p, y_p) be a coordinate chart containing the point $f(p)$ and let $V'_p = f(U_p) \cap V_p$. Then $f(p) \in V'_p \subset f(U_p)$, so $U'_p = f^{-1}(V'_p)$ is a neighborhood of p such that $f|_{U'_p} : U'_p \to V'_p$ is a diffeomorphism. Furthermore, $y_p|_{V'_p} \circ f|_{U'_p} : U'_p \to y_p|_{V'_p}(V'_p) \subset \mathbb{R}^n$ is a diffeomorphism, since it is the composition of diffeomorphisms.

Now, suppose $p, q \in M$ such that $U'_p \cap U'_q \neq \emptyset$. Then

$$(y_p|_{V'_p} \circ f|_{U'_p}) \circ (y_q|_{V'_q} \circ f|_{U'_q})^{-1} = y_p|_{V'_p} \circ (f|_{U'_p} \circ f|_{U'_q}^{-1}) \circ y_q|_{V'_q}^{-1} = y_p|_{V'_p} \circ y_q|_{V'_q}^{-1}$$

is a differentiable map from $y_q(V_p' \cap V_q')$ to $y_p(V_p' \cap V_q')$. Furthermore, since we can find such neighborhoods for each $p \in M$,

$$\bigcup_{p \in M} U_p' = M.$$

Therefore, $(U_p', y_p|_{V_p'} \circ f|_{U_p'})$ defines a differentiable structure on M. Finally, note that if $U_p' \cap U_q' \neq \emptyset$, then

$$\det(D((y_p|_{V_p'} \circ f|_{U_p'}) \circ (y_q|_{V_q'} \circ f|_{U_q'})^{-1})) = \det(D(y_p|_{V_p'} \circ y_q|_{V_q'}^{-1})) > 0$$

since N is orientable. Therefore, M is orientable.

On the other hand, if M is orientable and $f : M \to N$ is a bijective local diffeomorphism, then $f^{-1} : N \to M$ is a local diffeomorphism and we can apply the above result to see that N is orientable.

However, if f is not surjective, then N is not necessarily orientable. To see why, let N be any non-orientable manifold. Let (U, x) be a coordinate chart containing some point $p \in N$. Then

$$x : U \to x(U) \subset \mathbb{R}^n$$

is a diffeomorphism; let $M = x(U)$. Since $M = x(U) \subset \mathbb{R}^n$ is open, M is an orientable manifold and the map $x^{-1} : M \to N$ is a local diffeomorphism. However, despite M being orientable, N is not orientable.

Furthermore, if $f : M \to N$ is not injective, then the orientability of M does not guarantee the orientability of N. For example, consider the standard projection $\pi : S^2 \to \mathbb{RP}^2$. Let $M = S^2$ and $N = \mathbb{RP}^2$. Then M is orientable and N is not. Let (U_α, x_α) be an oriented atlas on M, let $p \in M$ and let (U, x) be a coordinate chart in this atlas containing p. Then, since $N = S^2/\{\pm 1\}$, we know that $(\pi_\alpha(U_\alpha), x_\alpha \circ \pi_\alpha|_{U_\alpha}^{-1})$ gives an atlas on N. Hence, both U and $\pi(U)$ are diffeomorphic to $x(U)$ and so $\pi|_U : U \to \pi(U)$ is a diffeomorphism. Since our choice of $p \in M$ was arbitrary, we see that such a neighborhood exists for all $p \in M$, so π is a local diffeomorphism. Therefore, we conclude that $f : M \to N$ being a local diffeomorphism and M being orientable does not necessarily mean that N is orientable.

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2

Let X be a vector field on a manifold M such that $X(p) \neq 0$ for all $p \in M$. Show that there exists a diffeomorphism $f : M \to M$ without any fixed points.

3

Let $f : SL(2,\mathbb{R}) \to \mathbb{R}$ where $f(A) = tr(A)$. Find the regular values of f. What is the diffeomorphism type of $f^{-1}(q)$ for q regular? What does $f^{-1}(q)$ look like for q singular?

Answer: Let $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{R})$. Then $ad - bc = 1$, so either a and d are both non-zero or b and c are both non-zero. Let

$$U_1 = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} | a, d \neq 0 \right\}, U_2 = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} | b, c \neq 0 \right\}.$$

Define the map $\phi_1 : U_1 \to \mathbb{R}^3$ by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto (a, b, d).$$

Since $b \neq 0$, and $ad - bc = 1$, $c = \frac{ad-1}{b}$, so we can define $\psi_1 : \mathbb{R}^3 - \{(a, 0, d)\} \to SL(2,\mathbb{R})$ by

$$(a, b, d) \mapsto \begin{pmatrix} a & b \\ \frac{ad-1}{b} & d \end{pmatrix}.$$

Then $\det(\psi_1(a, b, d)) = ad - \frac{ad-1}{b} b = ad - (ad - 1) = 1$ and

$$\phi_1 \circ \psi_1(a, b, d) = \phi_1 \begin{pmatrix} a & b \\ \frac{ad-1}{b} & d \end{pmatrix} = (a, b, d)$$

and

$$\psi_1 \circ \phi_1 \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \psi_1(a, b, d) = \begin{pmatrix} a & b \\ \frac{ad-1}{b} & d \end{pmatrix};$$

since $ad - bc = 1$, we see that $c = \frac{ad-1}{b}$. Therefore, ϕ_1 and ψ_1 are inverses; since we can view $SL(2,\mathbb{R})$ as a subset of $\mathbb{R}^4$ and ϕ_1 and ψ_1 are both clearly smooth when viewed as maps between $\mathbb{R}^4$ and $\mathbb{R}^3 - \{(a, 0, d)\}$, we conclude that, in fact, ϕ_1 is a homeomorphism of U_1 with $\mathbb{R}^3 - \{(a, 0, d)\}$.

Similarly, define the map $\phi_2 : U_2 \to \mathbb{R}^3$ by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto (a, b, c).$$

Then, since $a \neq 0$ and $ad - bc = 1$, $d = \frac{1+bc}{a}$, so we can define $\psi_2 : \mathbb{R}^3 - \{(0, b, c)\} \to SL(2,\mathbb{R})$ by

$$(a, b, c) \mapsto \begin{pmatrix} a & b \\ c & \frac{1+bc}{a} \end{pmatrix}.$$

Clearly, both ϕ_2 and ψ_2 are smooth. Furthermore, by a similar argument to that given above, $\phi_2 \circ \psi_2 = id$ and $\psi_2 \circ \phi_2 = id$, so $\phi_2 : U_2 \to \mathbb{R}^3 - \{(0, b, c)\}$ is a homeomorphism.

Now, $\phi_1(U_1) \cap \phi_2(U_2) = \mathbb{R}^3 - (\{(a, 0, d)\} \cup \{(0, b, c)\})$, so we need to check that $\phi_1 \circ \phi_2^{-1}$ is smooth on this overlap. To that end, let $(a, b, c) \in \mathbb{R}^3$ such that $a, b \neq 0$. Then

$$\phi_1 \circ \phi_2^{-1}(a,b,c) = \phi_1 \begin{pmatrix} a & b \\ c & \frac{1+bc}{a} \end{pmatrix} = \left(a, b, \frac{1+bc}{a}\right).$$

Since $a \neq 0$, this is a smooth map. Similarly,

$$\phi_2 \circ \phi_1^{-1}(a,b,c) = \phi_2 \begin{pmatrix} a & b \\ \frac{ad-1}{b} & c \end{pmatrix} = \left(a, b, \frac{ad-1}{b}\right).$$

Since $b \neq 0$, this is also a smooth map. Therefore, we conclude that (U_i, ϕ_i) gives an atlas on $SL(2,\mathbb{R})$ for $i = 1, 2$.

Now, consider the trace map. If $x_1 \neq 0$, then

$$tr \circ \phi_1^{-1}(x_1, x_2, x_3) = tr \begin{pmatrix} x_1 & x_2 \\ \frac{x_1 x_3 - 1}{x_2} & x_3 \end{pmatrix} = x_1 + x_3,$$

so

$$D(tr) = \left(\frac{\partial(tr \circ \phi_1^{-1})}{\partial x_1}, \frac{\partial(tr \circ \phi_1^{-1})}{\partial x_2}, \frac{\partial(tr \circ \phi_1^{-1})}{\partial x_3}\right) = (1, 0, 1),$$

which has rank 1, so none of these points is critical. On the other hand, if $b \neq 0$, then

$$tr \circ \phi_2^{-1}(x_1, x_2, x_3) = tr \begin{pmatrix} x_1 & x_2 \\ x_3 & \frac{1+x_2x_3}{x_1} \end{pmatrix} = x_1 + \frac{1+x_2x_3}{x_1},$$

so

$$D(tr) = \left(\frac{\partial(tr \circ \phi_2^{-1})}{\partial x_1}, \frac{\partial(tr \circ \phi_2^{-1})}{\partial x_2}, \frac{\partial(tr \circ \phi_2^{-1})}{\partial x_3}\right) = \left(1 - \frac{1+x_2x_3}{x_1^2}, \frac{x_3}{x_1}, \frac{x_2}{x_1}\right),$$

which almost always has rank 1. The only way this matrix can have rank 0 is if $x_2 = x_3 = 0$ and

$$\frac{1+x_2x_3}{x_1^2} = 1 \quad \Leftrightarrow \quad x_1^2 = 1 + x_2x_3 = 1,$$

i.e. $x_1 = \pm 1$. Since $\frac{1+x_2x_3}{\pm 1} = \frac{1}{\pm 1} = \pm 1$, we see that the only critical points of the trace map are

$$\begin{pmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{pmatrix}$$

so the critical values of the trace map are ±2.

♣

4

True or false?

(a): Every one form on $S^1 = \{x \in \mathbb{R}^2 | |x| = 1\}$ can be extended to a one form on $\mathbb{R}^2$.

Answer: True. Let $\omega \in \Omega^1(S^1)$. Define the map $g : \mathbb{R}^2 - \{0\} \to S^1$ by

$$g(x) = \frac{x}{|x|}.$$

Then g is certainly a smooth map, and so $g^*(\omega)$ is a 1-form on $\mathbb{R}^2-\{0\}$. Now, let $f : \mathbb{R}^2 \to \mathbb{R}$ be a smooth map such that $f(0,0) = 0$ and $f = 1$ outside $B(1/2)$, the ball of radius 1/2. Then $fg^*(\omega)$ is certainly a form on $\mathbb{R}^2 - \{0\}$, since $f : \mathbb{R}^2 - \{0\} \to \mathbb{R}$ is smooth. Furthermore, since $fg^*(\omega)$ is zero in a neighborhood of the origin, can be extended to a form η on $\mathbb{R}^2$ simply by defining $\eta = fg^*(\omega)$ away from the origin and $\eta = 0$ at the origin. Then, since $f = 1$ on S^1 and $g|_{S^1}$ is the identity, $\eta = fg^*(\omega) = \omega$ on S^1, so we see that η is an extension of the form ω to all of $\mathbb{R}^2$.

♣

(b): Every closed one form on S^1 can be extended to a closed one form on $\mathbb{R}^2$.

Answer: False. Let $\omega \in \Omega^1(S^1)$ be non-zero and closed. Suppose there existed an extension η of ω to all of $\mathbb{R}^2$ such that $d\eta = 0$. Let $B(1)$ be the ball of radius 1 in $\mathbb{R}^2$. Then, by Stokes' Theorem,

$$0 = \int_{B(1)} d\eta = \int_{\partial B(1)} \eta = \int_{S^1} \omega \neq 0,$$

as we showed on Homework 12, problem 3. From this contradiction, we conclude that there is no extension of the non-zero closed form $\omega \in \Omega^1(S^1)$ to a closed form on $\mathbb{R}^2$.

♣

5

Let M^n be compact and orientable. Show that if ω is an $(n-1)$-form, then $d\omega$ is zero at some point.

Proof. Since M has no boundary, we know, by Stokes' Theorem, that

$$\int_M d\omega = \int_{\partial M} \omega = \int_0 \omega = 0.$$

Let (U_α, x_α) be an atlas on M that yields an orientation. We may assume that, for each α, $x_\alpha(U_\alpha) = \mathbb{R}^n$ since $x_\alpha(U_\alpha) \subset \mathbb{R}^n$ is open and, hence, diffeomorphic to $\mathbb{R}^n$. Since M is compact, there exists a finite subset $U_1, \ldots, U_k$ of the U_α's such that

$$M = \bigcup_{i=1}^k U_i.$$

Let $\{\phi_i\}_{i=1}^k$ be a partition of unity subordinate to the U_i. Then

$$0 = \int_M d\omega = \sum_{i=1}^k \int_{U_i} \phi_i d\omega = \sum_{i=1}^k \int_{\mathbb{R}^n} (x_i^{-1})^*(\phi_i d\omega) = \int_{\mathbb{R}^n} \sum_{i=1}^k (x_i^{-1})^*(\phi_i d\omega).$$

Now, suppose $d\omega$ were everywhere non-zero. Suppose that $d\omega > 0$ at some $p \in M$ and $d\omega < 0$ at some $q \in M$. Then let $c : [0,1] \to M$ be a path from p to q. Then $d\omega$ must be zero at some point $c(s)$ for $s \in [0,1]$. Therefore, we see that $d\omega > 0$ or $d\omega < 0$ on all of M. Assume, without loss of generality, that $d\omega > 0$ on all of M. Then $d\omega \circ x_i^{-1} > 0$ on all of $\mathbb{R}^n$. Therefore, $(x_i^{-1})^*(\phi_i d\omega) \geq 0$ at all points and, furthermore, must be strictly positive at some point, since $\phi_i \neq 0$ and is a partition of unity (and, therefore, non-negative at every point). Thus, we see that

$$\int_{\mathbb{R}^n} \sum_{i=1}^{k} (x_i^{-1})^*(\phi_i d\omega) > 0.$$

From this contradiction, then, we conclude that $d\omega$ cannot be everywhere non-zero, and so $d\omega$ is zero at some point. □

6

Let $f : M \to N$ be smooth, M compact, and M and N orientable. Suppose there exists an open set $U \subset N$ such that $f : f^{-1}(U) \to U$ is a diffeomorphism. Show that N is compact.

Proof. Suppose N is not compact. Note that, since $f : f^{-1}(U) \to U$ is a diffeomorphism, M and N are manifolds of the same dimension; call this dimension n. Now, let ω be an n-form on M such that $\text{supp}(\omega) \subset W \subset f^{-1}(U)$ where (W, x) is a coordinate chart contained in $f^{-1}(U)$ and $\int_M \omega \neq 0$ (for example, take some volume form on W and multiply it by a smooth, non-negative function $g : M \to \mathbb{R}$ such that $\text{supp}(g) \subset W$ and $g = 1$ on some open set contained in W). Note that $\bar{W}$, the closure of W, is compact and contained in $f^{-1}(U)$, so $f(\bar{W})$ is compact and contained in U. Hence, $\bar{W}$ and $f(\bar{W})$ are compact n-manifolds with boundary. Then, since $f|_{\bar{W}}^{-1} : f(\bar{W}) \to \bar{W}$ is smooth, $\eta = (f|_{\bar{W}}^{-1})^*(\omega)$ is an n-form on U. Furthermore, η is zero on the boundary of $f(\bar{W})$, since $f|_{\bar{W}}^{-1}$ takes the boundary of $f(\bar{W})$ to the boundary of $\bar{W}$.

Now, extend η to a form η' on all of N by letting $\eta' = 0$ outside of $f(\bar{W})$. Since all n-forms are closed, $[\eta'] \in H^n(N)$. Now, since N is not compact, $H^n(N) = 0$, so $[\eta'] = 0$ in $H^n(N)$. Now, since f is smooth, f induces a homomorphism $f^* : H^n(N) \to H^n(M)$. Now consider $f^*(\eta')$. On $\bar{W}$,

$$f^*(\bar{W}) = f^*((f|_{\bar{W}}^{-1})^*(\omega)) = \omega.$$

Now, since $f^{-1}(f(\bar{W})) = \bar{W}$ (because $f : f^{-1}(U) \to U$ is a diffeomorphism) and since $\eta' = 0$ outside of $f(\bar{W})$, we see that $f^*(\eta') = (0 \circ f) dx^1 \wedge \ldots \wedge dx^n = 0$ outside of $\bar{W}$. However, this means that $f^*(\eta') = \omega$ both inside and outside of $\bar{W}$, so $f^*(\eta') = \omega$. Since M is compact, $H^n(M) = \mathbb{R}$ and

$$[\omega] \mapsto \int_M \omega$$

gives the isomorphism between $H^n(M)$ and $\mathbb{R}$. Now, recall that $\int_M \omega \neq 0$, so ω is a non-zero element of $H^n(M)$. Thus, $f^* : H^n(N) \to H^n(M)$ is a vector space homomorphism mapping zero (i.e. $[\eta']$)to a non-zero element (i.e. $[\omega]$). This is clearly impossible, so, from this contradiction, we conclude that, in fact, N is compact. $\square$

7

Define what one means by the product of two Lie algebras. Show that the Lie algebra of $SO(4)$ is the direct product of 2 Lie algebras each one of which is $(\mathbb{R}^3, \times)$ with its cross product.

Answer: Let $\mathfrak{g}$ and $\mathfrak{h}$ be two Lie algebras. Define the operation $[,] : \mathfrak{g} \times \mathfrak{h} \to \mathfrak{g} \times \mathfrak{h}$ by

$$[(X_1, Y_2), (X_2, Y_2)] = ([X_1, X_2], [Y_1, Y_2]).$$

Then we see that, for $(X, Y) \in \mathfrak{g} \times \mathfrak{h}$,

$$[(X, Y), (X, Y)] = ([X, X], [Y, Y]) = (0, 0)$$

since $\mathfrak{g}$ and $\mathfrak{h}$ are Lie algebras. Furthermore, if $(X_1, Y_1), (X_2, Y_2), (X_3, Y_3) \in \mathfrak{g} \times \mathfrak{h}$,

$$\begin{aligned}
&[[(X_1, Y_1), (X_2, Y_2)], (X_3, Y_3)] + [[(X_2, Y_2), (X_3, Y_3)], (X_1, Y_1)] + [[(X_3, Y_3), (X_1, Y_1)], (X_2, Y_2)] \\
&= [([X_1, X_2], [Y_1, Y_2]), (X_3, Y_3)] + [([X_2, X_3], [Y_2, Y_3]), (X_1, Y_1)] + [([X_3, X_1], [Y_3, Y_1]), (X_2, Y_2)] \\
&= ([[X_1, X_2], X_3], [[Y_1, Y_2], Y_3]) + ([[X_2, X_3], X_1], [[Y_2, Y_3], Y_1]) + ([[X_3, X_1]X_2], [[Y_3, Y_1], Y_2]) \\
&= ([[X_1, X_2], X_3] + [[X_2, X_3], X_1] + [[X_3, X_1], X_2], [[Y_1, Y_2], Y_3] + [[Y_2, Y_3], Y_1] + [[Y_3, Y_1], Y_2]) \\
&= (0, 0),
\end{aligned}$$

since the Jacobi identity holds in $\mathfrak{g}$ and $\mathfrak{h}$. Therefore, we see that $\mathfrak{g} \times \mathfrak{h}$ with bracket defined as above is a Lie algebra.

Now, let G and H be Lie groups with corresponding Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$. Then $G \times H$ is also a Lie group with identity (e_G, e_H). Let $(X_1, Y_1), (X_2, Y_2) \in T_{(e_G, e_H)}(G \times H)$. Then $(\widetilde{X_1, Y_1}) = (\widetilde{X_1}, \widetilde{Y_1})$ and $(\widetilde{X_2, Y_2}) = (\widetilde{X_2}, \widetilde{Y_2})$, so

$$\begin{aligned}
[(X_1, Y_1), (X_2, Y_2)] &= [(\widetilde{X_1, Y_1}), (\widetilde{X_2, Y_2})](e_G, e_H) \\
&= (\widetilde{X_1, Y_1})(\widetilde{X_2, Y_2})(e_G, e_H) - (\widetilde{X_2, Y_2})(\widetilde{X_1, Y_1})(e_G, e_H) \\
&= (\widetilde{X_1}, \widetilde{Y_1})(\widetilde{X_2}, \widetilde{Y_2})(e_G, e_H) - (\widetilde{X_1}, \widetilde{Y_1})(\widetilde{X_2}, \widetilde{Y_2})(e_G, e_H) \\
&= (\widetilde{X_1}\widetilde{X_2}(e_G), \widetilde{Y_1}\widetilde{Y_2}(e_H)) - (\widetilde{X_2}\widetilde{X_1}(e_G), \widetilde{Y_2}\widetilde{Y_1}(e_H)) \\
&= (\widetilde{X_1}\widetilde{X_2}(e_G) - \widetilde{X_2}\widetilde{X_1}(e_G), \widetilde{Y_1}\widetilde{Y_2}(e_H) - \widetilde{Y_2}\widetilde{Y_1}(e_H)) \\
&= ([\widetilde{X_1}, \widetilde{X_2}](e_G), [\widetilde{Y_1}\widetilde{Y_2}](e_H)) \\
&= ([X_1, X_2], [Y_1, Y_2]),
\end{aligned}$$

so we see that $\mathfrak{g} \times \mathfrak{h}$ as defined above is the Lie algebra for $G \times H$.

Now, turning to the case of $SO(4)$, we showed in Homework 8 Problem 5 that there exists a Lie group homomorphism $F : S^3 \times S^3 \to SO(4)$, which in turn induced a diffeomorphism $G : (S^3 \times S^3)/((x, y) \mapsto (-x, -y))$. Now, since $\{(1, 1), (-1, -1)\}$ is a discrete normal subgroup, $(S^3 \times S^3)/((x, y) \mapsto (-x, -y))$ is also a Lie group. Furthermore, since the group operation on this group is induced by the group operation on $S^3 \times S^3$ and F respects this group operation, we see that G is, in fact, a Lie group isomorphism.

Since the coordinate chart on $(S^3 \times S^3)/((x,y) \to (-x,-y))$ is simply given by $(\pi(U), x \circ \pi|_U^{-1})$ where π is the standard projection and (U,x) is an atlas on $S^3 \times S^3$, we see that these two Lie groups have the same tangent plane at the identity and, thus, the same Lie algebras. If $\mathfrak{s}^3$ is the Lie algebra of S^3, then our work above indicates that this Lie algebra is, in fact, $\mathfrak{s}^3 \times \mathfrak{s}^3$, which, given the above Lie group isomorphism, is also the Lie algebra of $SO(4)$, which we might call $\mathfrak{so}(4)$. Now, as we showed on Homework 8 Problem 2, $\mathfrak{s}^3$ is isomorphic to $(\mathbb{R}^3, \times)$, 3-space with its cross product. Therefore, we conclude that

$$\mathfrak{so}(4) \simeq (\mathbb{R}^3, \times) \times (\mathbb{R}^3, \times).$$

♣

8

Show that every 2 form on $\mathbb{R}^6$ has one of the following forms:

$$e_1 \wedge e_2, e_1 \wedge e_2 + e_3 \wedge e_4, e_1 \wedge e_2 + e_3 \wedge e_4 + e_5 \wedge e_6$$

with respect to some basis $e_1, \ldots, e_6$ (not necessarily the same basis for each form). Furthermore show that these 3 cases are mutually exclusive.

9

Compute the DeRham cohomology groups of the Klein bottle.

Answer: Let K denote the Klein bottle. Then, since K is connected, $H^0(K) = \mathbb{R}$. Also, since K is non-orientable, $H^2(\mathbb{R}) = 0$. Now, consider the two open subsets of K, U and V, indicated in the below picture:

Then both U and V are both cylinders and, as such, each is homotopic to S^1. Furthermore, $U \cap V$ consists of two extremely short cylinders and so is homotopy equivalent to the disjoint union of two copies of S^1. Now, form the Mayer-Vietoris sequence:

$$\begin{tikzcd}
0 \arrow[r] & H^0(K) \arrow[r] & H^0(U) \oplus H^0(V) \arrow[r] & H^0(U \cap V) \arrow[d, bend left] \\
H^2(K) & H^1(U \cap V) \arrow[l] & H^1(U) \oplus H^1(V) \arrow[l] & H^1(K) \arrow[l]
\end{tikzcd}$$

Note that $H^0(K) = \mathbb{R}$, $H^0(U) = H^0(V) = H^0(S^1) = \mathbb{R}$ and $H^1(U) = H^1(V) = H^1(S^1) = \mathbb{R}$. Furthermore, since $U \cap V$ has two components, an element of $H^0(U \cap V)$ is a constant function on each component (but can be different constants on each component) and since no zero form is exact, we see that $H^0(U \cap V) = \mathbb{R} \oplus \mathbb{R}$. Finally, the closed 1-forms on the disjoint union of two copies of S^1 consists of any two closed 1-forms on S^1 so, since $H^1(S^1) = \mathbb{R}$, we see that $H^1(U \cap V) = H^1(U \sqcup V) = \mathbb{R} \oplus \mathbb{R}$. Let us indicate these calculations on the sequence:

$$\begin{tikzcd}
0 \arrow[r] & H^0(K) \arrow[r] \arrow[d, phantom, "\wr"] & H^0(U) \oplus H^0(V) \arrow[r] \arrow[d, phantom, "\wr"] & H^0(U \cap V) \arrow[d, phantom, "\wr"] \arrow[dd, bend left] \\
 & \mathbb{R} & \mathbb{R} \oplus \mathbb{R} & \mathbb{R} \oplus \mathbb{R} \\
H^2(K) \arrow[d, equal] & H^1(U \cap V) \arrow[l] \arrow[d, phantom, "\wr"] & H^1(U) \oplus H^1(V) \arrow[l] \arrow[d, phantom, "\wr"] & H^1(K) \arrow[l] \\
0 & \mathbb{R} \oplus \mathbb{R} & \mathbb{R} \oplus \mathbb{R} &
\end{tikzcd}$$

Therefore, we see that $H^0(K) \to H^0(U) \oplus H^0(V)$ is injective, so its image is 1-dimensional and, hence, the kernel of $H^0(U) \oplus H^0(V) \to H^0(U \cap V)$ is 1-dimensional, meaning its image is also 1-dimensional. In turn, this implies that the kernel of $H^0(U \cap V) \to H^1(K)$ is 1-dimensional; since $H^0(U \cap V)$ is 2-dimensional, the image of this map is also 1-dimensional, and so the kernel of $H^1(K) \to H^1(U) \oplus H^1(V)$ is 1-dimensional.

On the other hand, since $H^1(U) \oplus H^1(V) \to H^1(U \cap V)$ must be surjective and both spaces are 2-dimensional, the kernel of this map must be trivial, and so the image of $H^1(K) \to H^1(U) \oplus H^1(V)$ is trivial. Therefore, since $H^1(K) \to H^1(U) \oplus H^1(V)$ has trivial image and 1-dimensional kernel, we see that $H^1(K)$ is 1-dimensional and, thus, must be isomorphic to $\mathbb{R}$ as a vector space. Therefore, we see that the complete cohomology of K is

$$H^i(K) = \begin{cases} \mathbb{R} & i = 0, 1 \\ 0 & i = 2 \end{cases}$$

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