GEOMETRY MIDTERM

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1

Let $f : M \to N$ and $g : N \to W$ be smooth. Show that $g \circ f : M \to W$ is smooth.

Proof. Let $p \in M$ and let $x : U \to \mathbb{R}^m$, $y : V \to \mathbb{R}^n$ and $z : A \to \mathbb{R}^k$ be coordinate charts on neighborhoods of p, $f(p)$ and $g \circ f(p)$, respectively. Then, since $f : M \to N$ is smooth, the map

$$y \circ f \circ x^{-1} : x(U) \to y(V)$$

is a smooth map from an open subset of $\mathbb{R}^m$ to an open subset of $\mathbb{R}^n$. Similarly, since $g : N \to W$ is smooth, the map

$$z \circ g \circ y^{-1} : y(V) \to z(A)$$

is a smooth map from an open subset of $\mathbb{R}^n$ to an open subset of $\mathbb{R}^k$. Now, we know that in Euclidean space the composition of smooth maps is smooth, so

$$(z \circ g \circ y^{-1}) \circ (y \circ f \circ x^{-1}) : x(U) \to z(A)$$

is smooth. However,

$$(z \circ g \circ y^{-1}) \circ (y \circ f \circ x^{-1}) = z \circ g \circ (y^{-1} \circ y) \circ f \circ x^{-1} = z \circ g \circ f \circ x^{-1} = z \circ (g \circ f) \circ x^{-1};$$

since this map is smooth, we see that, by defnition, $g \circ f$ is smooth. □

2

Show that there is no smooth map $f : S^1 \to S^1 \times S^1$ that is onto.

Proof. Let x_1 and x_2 be stereographic projection from the north and south poles, respectively, with corresponding open sets $U_1 = S^1 \backslash \{N\}$ and $U_2 = S^1 \backslash \{S\}$. Then $\{(U_1, x_1), (U_2, x_2)\}$ is an atlas on S^1. Furthermore, charts of the form $(U_i \times U_j, (x_i, x_j))$ form an atlas on $S^1 \times S^1$. Now, if there is some smooth, surjective map $f : S^1 \to S^1 \times S^1$, then the maps

$$(x_j, x_k) \circ f \circ x_i^{-1} : x_i(U_i) \to x_j(U_j) \times x_k(U_k)$$

must be smooth maps. Furthermore, since stereographic projection has all of $\mathbb{R}$ as its image,

$$g = (x_j, x_k) \circ f \circ x_i^{-1} : \mathbb{R} \to \mathbb{R} \times \mathbb{R} = \mathbb{R}^2$$

must have all of $\mathbb{R}^2$ except possibly one point (i.e. the image under $(x_j, x_k)^{-1} \circ f$ of either the relevant pole) in its image. In fact, a suitable rotation of

$S^1 \times S^1$ such that, if $i = 1$, the image of the north pole under f serves as the "pole" of the composite stereographic projection on $S^1 \times S^1$ means that

$$g : \mathbb{R} \to \mathbb{R}^2$$

is a surjective, smooth map. This is clearly impossible, though, so we see that, in fact, there is no smooth map $f : S^1 \to S^1 \times S^1$ that is onto. □

3

Show that the set of all $m \times n$ matrices of rank k is an embedded submanifold of $\mathbb{R}^{mn}$ of dimension $k(m+n-k)$.

Proof. First, let us consider the set M of all $m \times n$ matrices of rank $\leq k$. Now, let $A \in M$ and consider $(k+1)\times(k+1)$ minors of A. Specifically, each such minor must have zero determinant, else A would have to have rank at least $k+1$. If we label the minors by A_{ij} where the i and j are given by the row and column, respectively, of the top left entry in the minor, then we see that

$$\det A_{ij} = 0$$

for $1 \leq i \leq m-k$, $1 \leq j \leq n-k$. Hence, we have a system of $(m-k)(n-k)$ linear equations in mn unknowns, meaning there are

$$mn - (m-k)(n-k) = mn - (mn - kn - km - k^2) = k(n+m-k)$$

free variables, so we can naturally associate M with $\mathbb{R}^{k(n+m-k)}$ where the coordinate map is given by solutions of this system of linear equations in terms of the free variables (that is, simply selecting $k(n+m-k)$ entries in A that correspond to free variables in the linear system of equations and mapping A to the vector in $\mathbb{R}^{k(m+n-k)}$ determined by these entries). As constructed, this is clearly a continuous map, since this is essentially a projection of R^{mn} onto $R^{k(n+m-k)}$; the fact that it is bijective is also clear. The continuity of the inverse stems from the fact that we can solve the system of linear equations defined by the determinants of the minors purely by compositions of linear maps (corresponding to row-reduction, row interchange, etc.), which are, of course, continuous. Since this map defines global coordinates, we need not worry about coordinate interchange, and, since M inherits the paracompact and Hausdorff properties from its parent space, we see that M is a $k(m+n-k)$-dimensional manifold.

Define the map $g : M \to \mathbb{R}$ given by

$$g(A) = \max_{\substack{1 \leq i \leq m-k+1 \\ 1 \leq j \leq n-k+1}} \{|\det A_{ij}|\}$$

for $A \in M$ where the A_{ij} are $k \times k$ minors labeled in the way described above. Then the matrices in rank k in M will be precisely those mapped to positive reals; those matrices with rank $< k$ will be mapped to zero. Since the determinant maps are continuous, g is also continuous; since, for all $c \in \mathbb{R}^+$ (the positive reals),

$$\begin{pmatrix} c & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \end{pmatrix}$$

is mapped to c by g and the zero matrix maps to 0, we see that $g : M \to \mathbb{R}^+ \cup \{0\}$ is onto. Hence, $g^{-1}(\mathbb{R}^+)$ is precisely the set of matrices of rank k and, since g is continuous and $\mathbb{R}^+$ is open in $\mathbb{R}$, this corresponds to an open subset of $\mathbb{R}^{k(m+n-k)}$. From all this, then, we conclude that the $m \times n$ matrices of rank k form a manifold of dimension $k(m+n-k)$. The inclusion map provides the desired embedding into $\mathbb{R}^{mn}$. □

4

(a): Given two manifolds M^m and N^n, when is the product $M \times N$ orientable?

Answer: $M \times N$ is orientable if and only if M and N are both orientable. To see why, suppose, first, that M and N are both orientable. Then there exist atlases (U_α, x_α) and (V_α, y_α) on M and N, respectively, such that

$$\det(D(x_\alpha \circ x_\beta^{-1})) > 0$$

and

$$\det(D(y_\gamma \circ y_\delta^{-1})) > 0$$

for all α, β such that $U_\alpha \cap U_\beta \neq \emptyset$ and $V_\gamma \cap V_\delta \neq \emptyset$.

Now, $(U_\alpha \times V_\beta, (x_\alpha, y_\beta))$ gives an atlas on $M \times N$. Furthermore,

$$(x_\alpha, y_\beta) \circ (x_\gamma, y_\delta)^{-1}) = (x_\alpha \circ x_\gamma^{-1}, y_\beta \circ y_\delta^{-1}),$$

so

$$\begin{aligned} \det(D((x_\alpha, y_\beta) \circ (x_\gamma, y_\delta)^{-1})) &= \det(D(x_\alpha \circ x_\gamma^{-1}, y_\beta \circ y_\delta^{-1})) \\ &= \det \begin{pmatrix} D(x_\alpha \circ x_\gamma^{-1}) & 0 \\ 0 & D(y_\beta \circ y_\delta^{-1}) \end{pmatrix} \\ &= \det(D(x_\alpha \circ x_\gamma^{-1})) \det(D(y_\beta \circ y_\delta^{-1})) \\ &> 0 \end{aligned}$$

for all overlapping coordinate charts (x_α, y_β) and (x_γ, y_δ), so we see that $M \times N$ is orientable.

On the other hand, suppose $M \times N$ is orientable. Then, for each point $(p, q) \in M \times N$, there exists a neighborhood $W = U \times V$ and smooth maps $z_1, \ldots, z_{mn}$ such that $z_1(r, s), \ldots, z_{mn}(r, s)$ is an orientation on $T_{(r,s)}(M \times N)$ for all $(r,s) \in W$. Since $T_{(r,s)}(M \times N)$ is just a copy of R^{mn}, T_rM is a copy of $\mathbb{R}^m$, T_sN is a copy of $\mathbb{R}^n$ and $R_m \times R_n = \mathbb{R}^{mn}$, we can think of $T_{(r,s)}(M \times N)$ as being simply $T_rM \times T_sN$. Hence, orthogonally projecting $z_1(r,s), \ldots, z_{mn}(r,s)$ onto T_rM and T_sN yields smooth maps $\hat{z}_1, \ldots, \hat{z}_m$ and $\tilde{z}_1, \ldots \tilde{z}_n$, respectively, which give orientations on T_rM and T_sN.

Now, define the maps $a_i(r) = \hat{z}_i(r,q)$ and $b_i(s) = \tilde{z}_i(p,s)$. Since $\hat{z}_i$ smooth on $U \times V$, it is certainly also smooth on $U \times \{q\}$, so $a_i : U \to TM$ is smooth. Similarly, $b_i : V \to TN$ is smooth. Furthermore, as constructed, $a_1(r), \ldots, a_m(r)$ gives an orientation on T_rM for all $r \in U$ and $b_1(s), \ldots, b_n(s)$ gives an orientation on T_sN for all $s \in V$. Since our choice of $(p,q) \in M \times N$ was arbitrary, we see that, in fact, M and N must both be orientable if $M \times N$ is orientable.

♣

(b): Is $\mathbb{RP}^2 \times \mathbb{RP}^2$ or $\mathbb{RP}^2 \times S^2$ orientable?

Answer: No, neither is orientable. As we saw in class and proved in the homework, $\mathbb{RP}^n$ is orientable if and only if n is odd. Hence, $\mathbb{RP}^2$ is not orientable and, therefore, the result we proved in part (a) above demonstrates that neithr of these manifolds is orientable.

♣

5

Show that $T(x,y,z) = (1-z^2)y\frac{\partial}{\partial x} - (1-z^2)x\frac{\partial}{\partial y}$ defines a vector field on $S^2 = \{(x,y,z) | x^2+y^2+z^2 = 1\}$ and compute the flow of T on S^2. Can you describe it geometrically?

Proof. Let ϕ_1 be stereographic projection from the north pole and ϕ_2 be stereographic projection from the south pole. Then, as we've already seen, (U_i, ϕ_i) gives an atlas on S^2, where $U_1 = S^2 \backslash \{N\}$ and $U_2 = S^2 \backslash \{S\}$. Now, if we denote

$$T(x,y,z) = (1-z^2)y\frac{\partial}{\partial x} - (1-z^2)x\frac{\partial}{\partial y} = a_1(x,y,z)\frac{\partial}{\partial x} - a_2\frac{\partial}{\partial y}$$

then

$$\begin{aligned} a_1 \circ \phi_1^{-1}(u_1,u_2) &= a_1\left(\tfrac{4u_1}{u_1^2+u_2^2+4}, \tfrac{4u_2}{u_1^2+u_2^2+4}, \tfrac{u_1^2+u_2^2-4}{u_1^2+u_2^2+4}\right) \\ &= \left(1 - \left(\tfrac{u_1^2+u_2^2-4}{u_1^2+u_2^2+4}\right)^2\right)\left(\tfrac{4u_2}{u_1^2+u_2^2+4}\right) \\ &= \tfrac{4u_2[(u_1^2+u_2^2+4)^2-(u_1^2+u_2^2-4)^2]}{(u_1^2+u_2^2+4)^3} \\ &= \tfrac{4u_2(16u_1^2+16u_2^2)}{(u_1^2+u_2^2+4)^3} \end{aligned}$$

which, since the denominator is never zero, is smooth on $\mathbb{R}^2$. Similarly,

$$\begin{aligned} a_1 \circ \phi_2^{-1}(u_1, u_2) &= a_1\left(\tfrac{4u_1}{u_1^2+u_2^2+4}, \tfrac{4u_2}{u_1^2+u_2^2+4}, \tfrac{4-u_1^2-u_2^2}{4+u_1^2+u_2^2}\right) \\ &= \left(1 - \left(\tfrac{4-u_1^2-u_2^2}{u_1^2+u_2^2+4}\right)^2\right)\left(\tfrac{4u_2}{u_1^2+u_2^2+4}\right) \\ &= \tfrac{4u_2[(u_1^2+u_2^2+4)^2-(4-u_1^2-u_2^2)^2]}{(u_1^2+u_2^2+4)^3} \\ &= \tfrac{4u_2(16u_1^2+16u_2^2)}{(u_1^2+u_2^2+4)^3} \end{aligned}$$

which is, again, smooth. Similar calculations show that $a_2 \circ \phi_1^{-1}$ and $a_2 \circ \phi_2^{-1}$ are smooth, so we conclude that T does, indeed, define a smooth vector field on S^2.

Now, for the sake of convenience in writing, we can interpret the vector $T(x, y, z)$ in the tangent space of (x, y, z) as, simply, the vector

$$((1 - z^2)y, -(1 - z^2)x, 0)$$

in $\mathbb{R}^3$ translated to be based at the point (x, y, z). Now, consider some point $(x_0, y_0, z_0) \in S^2$ and suppose $c(t) = (c_1(t), c_2(t), c_3(t))$ is an integral curve through this point. Then $c(0) = (x_0, y_0, z_0)$ and

$$(c_1'(t), c_2'(t), c_3'(t)) = c'(t) = T \circ c(t) = ((1 - c_3^2(t))c_2(t), -(1 - c_3^2(t))c_1(t), 0).$$

Immediately, we see that $c_3'(t)$ is constant and, since $c_3(0) = z_0$, we conclude that $c_3(t) = z_0$ for all t. Hence, we can simplify the above equation to yield the following system:

$$\begin{aligned} c_1'(t) &= (1 - z_0^2)c_2(t) \\ c_2'(t) &= -(1 - z_0^2)c_1(t). \end{aligned}$$

Differentiating both sides yields

$$\begin{aligned} c_1''(t) &= (1 - z_0^2)c_2'(t) = -(1 - z_0^2)^2 c_1(t) \\ c_2''(t) &= -(1 - z_0^2)c_1'(t) = -(1 - z_0^2)^2 c_2(t). \end{aligned}$$

The fact that $c_1'' + (1 - z_0^2)^2 c_1 = 0$ implies that

$$c_1(t) = \alpha_1 \sin((1 - z_0^2)t) + \alpha_2 \cos((1 - z_0^2)t)$$

for some $\alpha_1, \alpha_2 \in \mathbb{R}$. Furthermore, since $c_2(t) = \frac{c_1'(t)}{1-z_0^2}$, we see that

$$c_2(t) = \alpha_1 \cos((1 - z_0^2)t) - \alpha_2 \sin((1 - z_0^2)t).$$

Plugging in the initial conditions, we see that

$$x_0 = c_1(0) = \alpha_2$$

and

$$y_0 = c_2(0) = \alpha_1$$

so, putting it all together, we see that

$$c(t) = (y_0 \sin((1-z_0^2)t)+x_0 \cos((1-z_0^2)t), y_0 \cos((1-z_0^2)t)-x_0 \sin((1-z_0^2)t), z_0).$$

Since our choice of (x_0, y_0, z_0) was arbitrary, we see that, for any $(x, y, z) \in S^2$, the flow through the point is given by

$$\phi_t(x,y,z) = (y\sin((1-z^2)t)+x\cos((1-z^2)t), y\cos((1-z^2)t)-x\sin((1-z^2)t), z).$$

Geometrically, the flow lines are simply the lines of latitude on the sphere.

□

6

Show that $\mathbb{CP}^1$ is diffeomorphic to S^2.

Proof. We prove this result by constructing an explicit diffeomorphism. From the first homework, we know that (U_i, x_i) gives an atlas on $\mathbb{CP}^1$, where

$$U_i = \{[z_1, z_2] \in \mathbb{CP}^1 | z_i \neq 0\}$$

and

$$x_1([z,w]) = w/z,$$
$$x_2([z,w]) = z/w.$$

Clearly, both x_1 and x_2 are maps from $\mathbb{CP}^1$ onto $\mathbb{C}$, since, for any $c \in \mathbb{C}$,

$$x_1(1,c) = c \qquad x_2(c,1) = c.$$

Furthermore, we know that (V_i, y_i) gives an atlas on S^2 where $V_1 = S^2 \backslash \{N\}$, $V_2 = S^2 \backslash \{S\}$ and y_1 and y_2 are stereographic projection from the north and south poles, respectively. Furthermore, both y_1 and y_2 are also onto $\mathbb{R}^2$, which we view as $\mathbb{C}$. Now, let us define our diffeomorphism $f : \mathbb{CP}^1 \to S^2$. For $z \neq 0$, let

$$f([z,w]) = y_1^{-1} \circ x_1([z,w]) = y_1^{-1}(w/z)$$

and, for $z = 0$, let $f([z,w]) = N = y_2^{-1}(0) = y_2^{-1} \circ x_2([z,w])$ (note that in $\mathbb{CP}^1$ there is only one point $[0,w]$, since, for all $w_1, w_2 \in \mathbb{C}$, $\frac{w_2}{w_1}(0, w_1) = (0, w_2)$).

Now, since y_1 and x_1 are coordinate charts, y_1^{-1} and x_1 are differentiable, so, by problem 1 above, their composition f is differentiable away from the possible bad point $[0,w]$. Furthermore, y_1 and x_1^{-1} are also differentiable, so $x_1^{-1} \circ y_1$ is differentiable away from the north pole of S^2. Therefore, we see that f is differentiable with differentiable inverse at all points except possibly $[0,w]$. Now, as $z \to 0$, $w/z \to \infty$ and so $f([z,w]) \to N$; since we defined $f([0,w]) = N$, we see that f is, in fact, continuous at $[0,w]$ with continuous inverse.

For z near 0, $x_2([z,w]) = z/w$, so

$$y_2 \circ f \circ x_2^{-1}(z/w) = y_2 \circ y_1^{-1} \circ x_1 \circ x_2^{-1}(z/w) = y_2 \circ y_1^{-1} \circ x_1([z/w, 1]) = y_2 \circ y_1^{-1}(w/z),$$

which will be close to the origin, since w/z has very large norm, meaning y_1^{-1} maps it near the north pole, points near which are mapped near the origin by y_2. In fact, we see that $y_2 \circ f \circ x_2^{-1}(z/w)$ approaches the identity map as z approaches 0 and, since

$$y_2 \circ f \circ x_2^{-1}(0) = y_2 \circ f([0,1]) = y_2(N) = 0$$

we see that it is the identity map at the origin. Now, again using our result in problem 1, we know that $y_2 \circ y_1^{-1} \circ x_1 \circ x_2^{-1}$ is differentiable (as is its inverse) and the identity map is certainly differentiable, so we conclude that f is, in fact differentiable at $[0, w]$. Since the inverse map will also give the identity at N, we see that f has differentiable inverse at $[0, w]$ as well.

Therefore, we conclude that, in fact, f is a diffeomorphism and that, therefore, $\mathbb{CP}^1$ and S^2 are diffeomorphic.

□

7

Show that $\{M \in SO(3) | M = M^t, M \neq Id\}$ is a submanifold of $SO(3)$. Is this true if I include $M = Id$? Can you identify the surface?

Proof. Let $N = \{M \in SO(3) | M = M^t, M \neq Id\}$. Note, first off, that all elements of N are orthogonally diagonalizable since they are symmetric. Hence, for all $B \in N$,

$$B = PDP^{-1}$$

for orthogonal P and diagonal D. The non-zero elements of D are simply the eigenvalues of B; since B is orthogonal these eigenvalues must have norm 1, since B is symmetric they must be real and since $1 = \det B$, which is just the product of the eigenvalues, we see that each non-zero element of D must be ± 1, with an even number of -1's. If all eigenvalues of B are 1, then $D = Id$ and so

$$B = PDP^{-1} = PIdP^{-1} = PP^{-1} = Id,$$

which is not in N; hence, we conclude that B must have -1 as an eigenvalue with positive, even multiplicity. Namely, B has eigenvalues 1 and -1, with 1 of multiplicity 1 and -1 of multiplicity 2.

Now, define the map $f : SO(3) \to N$ by

$$f : A \mapsto A \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} A^{-1}.$$

Note, first, that this map is well-defined as, if

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix},$$

then

$$f(A) = \begin{pmatrix} a_{11}^2 - a_{12}^2 - a_{13}^2 & a_{11}a_{21} - a_{12}a_{22} - a_{13}a_{23} & a_{11}a_{31} - a_{12}a_{32} - a_{13}a_{33} \\ a_{11}a_{21} - a_{12}a_{22} - a_{13}a_{23} & a_{21}^2 - a_{22}^2 - a_{23}^2 & a_{21}a_{31} - a_{22}a_{32} - a_{23}a_{33} \\ a_{11}a_{31} - a_{12}a_{32} - a_{13}a_{33} & a_{21}a_{31} - a_{22}a_{32} - a_{23}a_{33} & a_{31}^2 - a_{32}^2 - a_{33}^2 \end{pmatrix}$$

which is symmetric. Furthermore, note that this map is continuous when viewed as a map in $\mathbb{R}^9$ and thus, since $SO(3)$ inherits the subspace topology, is continuous on $SO(3)$. Furthermore, f is surjective, since for any $B \in N$, B can be orthogonally diagonalized as

$$B = PDP^{-1}$$

with $D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$ as we saw above. Furthermore, for $A, C \in SO(3)$, if $f(A) = f(C) = B$, then, since the columns of A and C are simply the eigenvectors corresponding to the appropriate eigenvalues of B, it must be the case that $A = C$, so f is injective. Finally, since eigenvectors vary continuously, the inverse of f is also continuous. Thus, f is a bijective continuous map with continuous inverse, which is to say a homeomorphism. Since $N \subset SO(3)$ and N is homeomorphic to $SO(3)$ by way of f, we see that N must be an open set of $SO(3)$ and, therefore, a submanifold of $SO(3)$.

Note that we cannot expand f or f^{-1} in any way to include Id, since every vector is an eigenvector of the identity. In fact, note that the only eigenvalue of Id is 1, whereas all elements of N have -1 as an eigenvalue; since the eigenvalues of a matrix vary continuously with the matrix, we see that Id is in fact disconnected from N. As such, we see that $N \cup \{Id\}$ is not a manifold, as any proposed atlas would have to cover Id with some open set U and the image of U under a coordinate chart would be either a single point or the union of an open set with an isolated point, neither of which is an open set in euclidean space. □

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