GEOMETRY FINAL

CLAY SHONKWILER

1

If V is an n-dimensional euclidean vector space, show that $*: \Lambda^p \to \Lambda^{n-p}$ is well defined by $*(v_1 \wedge v_2 \wedge \cdots \wedge v_p) = v_{p+1} \wedge \cdots \wedge v_n$ for any oriented orthonormal basis $v_1, \ldots, v_n$.

2

Show that $** = (-1)^{p(n-p)} id$ and that $*$ is an isometry.

3

Show that the 2 inner products on $\Omega^p(M)$ given by $\langle \alpha, \beta \rangle = \int_M \alpha \wedge *\beta$ and $\langle \alpha, \beta \rangle = \int_M \langle \alpha, \beta \rangle dvol_g$ are the same.

4

Show that on $\mathbb{R}^n$ with its usual inner product $\triangle(f) = -\sum_i \partial^2 f / \partial x_i^2$ for any function f. What is $\triangle(\omega)$ for a p-form ω?

Proof. Let f be a 0-form on $\mathbb{R}^n$. Then

$$\begin{aligned}
\triangle f &= d\delta f + \delta df \\
&= d * d * f + *d * df \\
&= d * d(f dx^1 \wedge \ldots \wedge dx^n) + *d * \sum_{i=1}^{n} \frac{\partial f}{\partial x^i} dx^i \\
&= d * (df \wedge dx^1 \wedge \ldots \wedge dx^n) + *d \sum_{i=1}^{n} (-1)^{i-1} \frac{\partial f}{\partial x^i} dx^1 \wedge \ldots \wedge \widehat{dx^i} \wedge \ldots \wedge dx^n \\
&= 0 + * \sum_{i=1}^{n} (-1)^{i-1} \frac{\partial^2 f}{\partial {x^i}^2} dx^i \wedge dx^1 \wedge \ldots \wedge \widehat{dx^i} \wedge \ldots \wedge dx^n \\
&= * \sum_{i=1} (-1)^{i-1} (-1)^{i-1} \frac{\partial^2 f}{\partial {x^i}^2} dx^1 \wedge \ldots \wedge dx^n \\
&= \sum_{i=1}^{n} \frac{\partial^2 f}{\partial {x^i}^2}.
\end{aligned}$$

Suppose $\alpha = f dx^{\sigma(1)} \wedge \ldots \wedge dx^{\sigma(p)}$ is a p-form, where $1 \leq \sigma(1) < \ldots < \sigma(p) \leq n$, $1 \leq \sigma(p+1) < \ldots < \sigma(n) \leq n$. Let $\epsilon = \operatorname{sgn}(\sigma)$. Then

$$\begin{aligned}
\triangle\alpha &= (d\delta + \delta d)\alpha \\
&= d * d * f dx^{\sigma(1)} \wedge \ldots \wedge dx^{\sigma(p)} + *d * df dx^{\sigma(1)} \wedge \ldots \wedge dx^{\sigma(p)} \\
&= d * d(-1)^{\epsilon} f dx^{\sigma(p+1)} \wedge \ldots \wedge dx^{\sigma(n)} + *d * \sum_{i=p+1}^{n} \frac{\partial f}{\partial x^{\sigma(i)}} dx^{\sigma(i)} \wedge dx^{\sigma(1)} \wedge \ldots \wedge dx^{\sigma(p)} \\
&= d * \sum_{i=1}^{p} (-1)^{\epsilon} \frac{\partial f}{\partial x^{\sigma(i)}} dx^{\sigma(i)} \wedge dx^{\sigma(p+1)} \wedge \ldots \wedge dx^{\sigma(n)} + *d \sum_{i=p+1}^{n} (-1)^{-\epsilon} \frac{\partial f}{\partial x^{\sigma(i)}} dx^{\sigma(p+1)} \wedge \ldots \wedge \\
&= d \sum_{i=1}^{p} (-1)^{\epsilon} \frac{\partial f}{\partial x^{\sigma(i)}} dx^{\sigma(1)} \wedge \ldots \wedge \widehat{dx^{\sigma(i)}} \wedge \ldots \wedge dx^{\sigma(p)} + * \sum_{i=p+1}^{n} (-1)^{-\epsilon} \left(\frac{\partial^2 f}{\partial {x^{\sigma(i)}}^2} dx^{\sigma(i)} + \sum_{j=1}^{p} \right. \\
&= \sum_{i=1}^{p} (-1)^{\epsilon} \left(\frac{\partial^2 f}{\partial {x^{\sigma(i)}}^2} dx^{\sigma(i)} + \sum_{j=p+1}^{n} \frac{\partial^2 f}{\partial x^{\sigma(j)} \partial x^{\sigma(i)}} dx^{\sigma(j)} \right) \wedge dx^{\sigma(1)} \wedge \ldots \wedge \widehat{dx^{\sigma(i)}} \wedge \ldots \wedge dx^{\sigma(p)} +
\end{aligned}$$

□

5

Check that $\langle d\alpha, \beta \rangle = \langle \alpha, \delta\beta \rangle$ where $\delta\beta = (-1)^{n(p+1)+1} * d * \beta$.

6

Show that a harmonic form is closed and co-closed, that an exact harmonic form is 0, and that $\triangle$ commutes with $*$.

Proof. Suppose α is a harmonic form; that is, $\triangle\alpha = 0$. Then

$$\begin{aligned}
0 = \langle \triangle\alpha, \alpha \rangle &= \langle d\delta\alpha + \delta d\alpha, \alpha \rangle \\
&= \langle d\delta\alpha, \alpha \rangle + \langle \delta d\alpha, \alpha \rangle \\
&= \langle \delta\alpha, \delta\alpha \rangle + \langle d\alpha, d\alpha \rangle.
\end{aligned}$$

This, then, implies that $\delta\alpha = d\alpha = 0$, since $\langle , \rangle$ is positive definite.

Now, suppose $\triangle\alpha = 0$ and $\alpha = d\omega$ for some form ω. □

7

Compute the fundamental group of:

- $\mathbb{R}^n$ minus a point.

  Answer: Since $\mathbb{R}^n$ minus a point is homeomorphic to $\mathbb{R}^n - \{0\}$, we may as well assume the deleted point is the origin. Let S^n be the unit sphere centered at the origin. Define the map $H : [0,1] \times \mathbb{R}^n - \{0\} \to S^{n-1}$ by

  $$x \mapsto t\frac{x}{|x|} + (1-t)x.$$

Then for all $x \in \mathbb{R}^n - \{0\}$, $H(0,x) = x$ and

$$|H(1,x)| = \left| \frac{x}{|x|} \right| = \frac{|x|}{|x|} = 1,$$

so $H(1,x) \in S^{n-1}$. Furthermore, if $x \in S^{n-1}$, then $|x| = 1$ and so

$$H(t,x) = t\frac{x}{|x|} + (1-t)x = tx + (1-t)x = x,$$

so H is the identity map on S^{n-1}. Now, H is continuous, so we see that S^{n-1} is a deformation retract of $R^n - \{0\}$, so

$$\pi_1(\mathbb{R}^n - \{0\}) = \pi_1(S^{n-1}) = \left\{ \begin{array}{ll} \mathbb{Z} & n = 2 \\ 1 & n > 2 \end{array} \right.$$

Finally, if $n = 1$, then $\mathbb{R}^n - \{0\}$ is not connected, consisting of two simply connected components. Since each component is simply connected, its fundamental group is trivial.

♣

- $\mathbb{R}^n$ minus a line.

Answer: Clearly, this only make sense when $n \geq 2$. Now, if $n = 2$, then $\mathbb{R}^2$ minus a line consists of two components, each of which is simply connected, and so the fundamental group of each component is trivial.

If $n > 2$, let X denote the x_1-axis. Then $\mathbb{R}^n$ minus a line is homeomorphic to $\mathbb{R}^n - X$, so we may as well assume the deleted line is X. Let $Y = \{(0, x_2, \ldots, x_n) \in \mathbb{R}^n - X\}$. Define $H : [0,1] \times \mathbb{R}^n - X \to Y$ by

$$(x_1, \ldots, x_n) \mapsto ((1-t)x_1, x_2, \ldots, x_n).$$

Then H is certainly continuous. Furthermore, for all $(x_1, \ldots, x_n) \in \mathbb{R}^n - X$,

$$H(0, (x_1, \ldots, x_n)) = (x_1, \ldots, x_n)$$

and

$$H(1, (x_1, \ldots, x_n)) = (0, x_2, \ldots, x_n) \in Y.$$

Finally, if $(0, x_2, \ldots, x_n) \in Y$, then

$$H(t, (0, x_2, \ldots, x_n)) = (0, x_2, \ldots, x_n),$$

so H is a deformation retract of $\mathbb{R}^n - X$. Now, note that Y is homeomorphic to $\mathbb{R}^{n-1} - \{0\}$, so we see that

$$\pi_1(\mathbb{R}^n - X) = \pi_1(\mathbb{R}^{n-1} - \{0\}) = \left\{ \begin{array}{ll} \mathbb{Z} & n = 3 \\ 1 & n > 3 \end{array} \right.$$

by our work above.

♣

- $\mathbb{R}^n$ minus a circle.

  Answer: Again, this only makes sense when $n \geq 2$. If $n = 2$, then $\mathbb{R}^2$ minus a circle consists of two components. The component inside the circle is simply connected and, hence, has trivial fundamental group. The component outside the circle is homotopic to $\mathbb{R}^2$ minus a point, and thus has fundamental group isomorphic to $\mathbb{Z}$.

  If $n > 2$
- $\mathbb{R}^n$ minus a line and a circle. There are two possible spaces. Show that they are not homotopy equivalent.

8

Show that every covering is a local homeomorphism and find a local homeomorphism that is not a covering.

Proof. Let $\pi : X \to Y$ be a covering. Let $p \in X$ and let $q = \pi(p) \in Y$. Then, since π is a covering map, there exists a neighborhood U of q such that

$$\pi^{-1}(q) = \bigcup_{\alpha \in A} V_\alpha$$

for some indexing set A where $V_\alpha \cap V_\beta = \emptyset$ for $\alpha \neq \beta$ and $\pi|_{V_\alpha} : V_\alpha \to U$ is a homeomorphism for all $\alpha \in A$. Then, since $p \in \pi^{-1}(q)$, $p \in V_\alpha$ for exactly one α. Furthermore, since π is continuous, V_α is open and

$$\pi|_{V_\alpha} : V_\alpha \to U$$

is a homeomorphism. Since our choice of the point p was arbitrary, we see that there exists such a neighborhood for any point in X, so π is a local homeomorphism.

□

9

Is $f : \mathbb{C} \to \mathbb{C}$ given by $f(z) = e^z$ a covering? If not, can I modify this to make it into a covering?

Answer: Let $z = x + iy \in \mathbb{C}$. Then

$$f(z) = e^z = e^{x+iy} = e^x e^{iy}.$$

Since $e^x > 0$ for all $x \in \mathbb{R}$ and e^{iy} is a point on the unit circle centered at the origin, we see that $f(z)$ lies on a circle of radius $e^x > 0$; specifically, $f(z) \neq 0$, so $f : \mathbb{C} \to \mathbb{C}$ cannot be a covering.

DRL 3E3A, University of Pennsylvania
E-mail address: `shonkwil@math.upenn.edu`