GEOMETRY HW 10

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1

Show that every continuous map $f : \mathbb{CP}^{2n} \to \mathbb{CP}^{2n}$ has a fixed point. How about $\mathbb{CP}^{2n+1}$?

Proof. Let $f : \mathbb{CP}^{2n} \to \mathbb{CP}^{2n}$ be continuous. Then $f_* : H_i(\mathbb{CP}^{2n}) \to H_i(\mathbb{CP}^{2n})$ and $f^* : H^i(\mathbb{CP}^{2n}) \to H^i(\mathbb{CP}^{2n})$ for all i. Now,

$$H_i(\mathbb{CP}^{2n}) = H^i(\mathbb{CP}^{2n}) = \begin{cases} \mathbb{Z} & i \text{ even} \\ 0 & i \text{ odd.} \end{cases}$$

Hence, in odd dimensions, $f_* \equiv 0$ and $f^* \equiv 0$; in even dimensions, f_* and f^* are just multiplication by some integer. Now, if $\alpha \in H^{2i}(\mathbb{CP}^{2n})$ and $\sigma \in C_{2i}(\mathbb{CP}^{2n})$, then

$$f^*(\alpha)(\sigma) = \alpha(f \circ \sigma) = \alpha(f_*(\sigma)),$$

so we see that f_* and f^* are multiplication by the same integer in each dimension. Hence, the Lefshetz number

$$\begin{aligned}\tau(f) &= \sum_{k=0}^{4n} (-1)^k \mathrm{tr}(f_* : H_k(\mathbb{CP}^{2n}) \to H_k(\mathbb{CP}^{2n})) \\ &= \sum_{k=0}^{4n} (-1)^k \mathrm{tr}(f^* : H^k(\mathbb{CP}^{2n}) \to H^k(\mathbb{CP}^{2n})) \\ &= \sum_{k=0}^{2n} (-1)^{2k} \mathrm{tr}(f^* : H^{2k}(\mathbb{CP}^{2n}) \to H^{2k}(\mathbb{CP}^{2n})) \\ &= \sum_{k=0}^{2n} \mathrm{tr}(f^* : H^{2k}(\mathbb{CP}^{2n}) \to H^{2k}(\mathbb{CP}^{2n})),\end{aligned}$$

since $f^* \equiv 0$ in even dimensions. Also, note that the trace of f^* is simply whatever integer it multiplies $1 \in \mathbb{Z} \simeq H^{2k}(\mathbb{CP}^{2n})$ by (where $1 \in \mathbb{Z}$ corresponds to the generator of $H^{2k}(\mathbb{CP}^{2n})$).

Now, f^* is a ring homomorphism and $H^*(\mathbb{CP}^{2n}) = \mathbb{Z}[a]/(z^{n+1})$ where a has grade 2. Hence, if $[\gamma]$ is the generator of $H^{2k}(\mathbb{CP}^{2n})$, then

$$[\gamma] = \underbrace{[\alpha] \cup \cdots \cup [\alpha]}_{k \text{ copies}}$$

where α is the generator of $H^2(\mathbb{CP}^{2n})$. Therefore,

$$f^*(\gamma) = f^*(\alpha \cup \cdots \cup \alpha) = f^*(\alpha) \cup \cdots \cup f^*(\alpha),$$

so $\operatorname{tr}(f^* : H^{2k}(\mathbb{CP}^{2n}) \to H^{2k}(\mathbb{CP}^{2n})) = \left[\operatorname{tr}(f^* : H^2(\mathbb{CP}^{2n}) \to H^2(\mathbb{CP}^{2n}))\right]^k$. Hence, if $\operatorname{tr}(f^* : H^2(\mathbb{CP}^{2n}) \to H^2(\mathbb{CP}^{2n})) = A$, then

$$\tau(f) = \sum_{k=0}^{2n} \operatorname{tr}(f^* : H^{2k}(\mathbb{CP}^{2n}) \to H^{2k}(\mathbb{CP}^{2n})) = \sum_{k=0}^{2n} A^k.$$

Thus, the only way we could possibly have $\tau(f) = 0$ is when $A = -1$. However, since $(-1)^0 = 1$ and $(-1)^{2n} = 1$, we see that, even if $A = -1$, $\tau(f) = 1 \neq 0$, so, by the Lefshetz theorem, f has a fixed point.

A similar argument demonstrates that, in the case of $f : \mathbb{CP}^{2n+1} \to \mathbb{CP}^{2n+1}$,

$$\tau(f) = \sum_{k=0}^{2n+1} A^k$$

where $A = f^*(\alpha)$ for $[\alpha]$ a generator of $H^2(\mathbb{CP}^{2n+1})$. Again, the only possible problem arises when $A = -1$ (that is, when $f^*(\alpha) = -\alpha$); in this case, this means that

$$\tau(f) = \sum_{k=0}^{2n+1} (-1)^k = (1-1) + (1-1) + \cdots + (1-1) = 0;$$

in all other cases, $\tau(f) \neq 0$ and so f has a fixed point. Now, to see that there is a map with no fixed points, consider $f : \mathbb{C}^{2n+2} \to \mathbb{C}^{2n+2}$ given by

$$(z_1, \ldots, z_{2n+2}) \mapsto (\bar{z}_2, -\bar{z}_1, \bar{z}_4, \ldots, \bar{z}_{2n+2}, -\bar{z}_{2n+1}).$$

Then f is continuous and has no fixed points; consider the induced map $\bar{f} : \mathbb{CP}^{2n+1} \to \mathbb{CP}^{2n+1}$. Then

$$f(z_1, \ldots, z_{2n+2}) = (z_1, \ldots, z_{2n+2})$$

implies that

$$\begin{aligned} \lambda \bar{z}_2 &= z_1, \\ -\lambda \bar{z}_1 &= z_2, \\ &\vdots \\ \lambda \bar{z}_{2n+2} &= z_{2n+1} \\ -\lambda \bar{z}_{2n+1} &= z_{2n+2} \end{aligned}$$

for some $\lambda \in \mathbb{C}$. Hence,

$$z_2 = -\lambda \bar{z}_1 = -\lambda \overline{\lambda \bar{z}_2} = -|\lambda|^2 z_2,$$

which implies that $|\lambda|^2 = -1$, a clear impossibility. Therefore, f has no fixed point. □

2

Let A be two unlinked circles in $\mathbb{R}^3$ and B two simply linked circles. Show that the complements have isomorphic cohomology groups but different ring structures.

Proof. From Hatcher, pp. 46-47, $S^1 \vee S^1 \vee S^2 \vee S^2$ is a deformation retract of A and $S^2 \vee T^2$ is a deformation retract of B. Now,

$$H^0(A) = H^0(S^1 \vee S^1 \vee S^2 \vee S^2) = \mathbb{Z}$$
$$H^1(A) = H^1(S^1 \vee S^1 \vee S^2 \vee S^2) = H^1(S^1) \oplus H^1(S^1) \oplus H^1(S^2) \oplus H^1(S^2) = \mathbb{Z} \oplus \mathbb{Z}$$
$$H^2(A) = H^1(S^1 \vee S^1 \vee S^2 \vee S^2) = H^2(S^1) \oplus H^2(S^1) \oplus H^2(S^2) \oplus H^2(S^2) = \mathbb{Z} \oplus \mathbb{Z}$$

and

$$H^0(B) = H^0(S^2 \vee T^2) = \mathbb{Z}$$
$$H^1(B) = H^1(S^2 \vee T^2) = H^1(S^2) \oplus H^1(T^2) = \mathbb{Z} \oplus \mathbb{Z}$$
$$H^2(B) = H^1(S^2 \vee T^2) = H^2(S^2) \oplus H^2(T^2) = \mathbb{Z} \oplus \mathbb{Z},$$

so A and B have the same cohomology groups. Now, if $[\alpha]$ is a generator of $H^1(A)$, then $[\alpha]$ is a generator of $H^1(S^1)$ for one of the S^1's in the deformation retract of A. Then, for such an α,

$$[\alpha] \cup [\alpha] \in H^2(S^1) = 0,$$

so $[\alpha] \cup [\alpha] = 0$. Since this was the only possible non-trivial product in the ring structure induced by the cup product on $H^*(A)$, we see that $H^*(A)$ has a trivial ring structure.

On the other hand, let $[\alpha]$ and $[\beta]$ be generators of $H^1(B)$. Then $[\alpha]$ and $[\beta]$ can be viewed as generators of $H^1(T^2)$. Furthermore, in the cup product structure on $H^*(T^2)$,

$$[\alpha] \cup [\beta] = [\gamma]$$

where $[\gamma] \neq 0$ is the generator of $H^2(T^2)$, which, in turn, is a generator of $H^2(B)$. Therefore, $H^*(B)$ has a non-trivial cup product structure, and so, since cup product structure is a homotopy invariant, A and B are not homotopy equivalent. □

3

Show that under the natural inclusion $\mathbb{RP}^\infty \hookrightarrow \mathbb{CP}^\infty$, the map in cohomology (in $\mathbb{Z}$ coefficients) is onto in every even dimension.

Proof. Consider the natural inclusion $i : \mathbb{RP}^\infty \hookrightarrow \mathbb{CP}^\infty$. Now, we know that $H^*(\mathbb{RP}^\infty) = \mathbb{Z}/2[\alpha]$ and $H^*(\mathbb{CP}^\infty) = \mathbb{Z}[\beta]$ where α is of grade 1 and β is of grade 2. Specifically, if $[\gamma]$ is a generator of $H^{2k}(\mathbb{CP}^\infty)$, then

$$[\gamma] = \underbrace{[\alpha] \cup \cdots \cup [\alpha]}_{k \text{ times}}$$

Also, $f^*(\gamma) = f^*(\alpha \cup \cdots \cup \alpha) = f^*(\alpha) \cup \cdots \cup f^*(\alpha)$ for any continuous $f : \mathbb{RP}^\infty \to \mathbb{CP}^\infty$. On the other hand, if $[\eta]$ is a generator of $H^j(\mathbb{RP}^\infty)$, then

$$[\eta] = [\beta] \cup \cdots \cup [\beta]$$

for $[\beta]$ a generator of $H^1(\mathbb{RP}^\infty)$. Therefore, if $f^*(\alpha) = \beta^2$, then, if $[\gamma]$ is a generator of $H^{2k}(\mathbb{CP}^\infty)$ and $[\eta]$ a generator of $H^{2k}(\mathbb{RP}^\infty)$, then

$$f^*(\gamma) = f^*(\alpha) \cup \cdots \cup f^*(\alpha) = \beta^2 \cup \cdots \cup \beta^2 = \beta^{2k} = \eta,$$

so f^* is surjective in even dimensions. The only other alternative is that $f^*(\alpha) = 0$, in which case $f^*(\gamma) = 0$ where $[\gamma]$ is the generator of $H^{2k}(\mathbb{CP}^\infty)$. In other words, either f^* is surjective in every even dimension, or f^* is the zero map for all positive dimensions, and it all hinges on whether $f^*(\alpha)$ is non-zero.

Now, consider the map $\mathbb{RP}^2 \xrightarrow{j} \mathbb{CP}^2 \setminus \{(1 : i : 0)\} \xrightarrow{\pi} \mathbb{CP}^1$, where j is the natural inclusion and

$$\pi : (x_0 : x_1 : x_2) \mapsto (x_0 + ix_1 : x_2).$$

Since $(1 : i : 0) \notin \operatorname{im} j$, j and π are well-defined and certainly continuous. Now, we know that (U_i, ψ_i) with $U_i = \{(x_0 : \ldots : x_{i-1} : 1 : x_{i+1} : \ldots : x_n)\}$ and $\psi_i : U_i \to \mathbb{R}^2$ given by

$$(x_0 : \ldots : 1 : \ldots : x_n) \mapsto (x_0, \ldots, \widehat{x_i}, \ldots, x_n)$$

define coordinate charts on $\mathbb{RP}^n$. Similarly, (V_i, ϕ_i) where $V_i = \{(x_0 + iy_0 : \ldots : 1 : \ldots : x_n + iy_n)\}$ and $\phi_i : V_i \to \mathbb{R}^{2n}$ given by

$$(x_0 + iy_0 : \ldots : 1 : \ldots : x_n + iy_n) \mapsto (x_0, y_0, \ldots, \widehat{x_i}, \widehat{y_i}, \ldots, x_n, y_n)$$

give coordinate charts on $\mathbb{CP}^n$ for all n. Now, for $(x, y) \in \mathbb{R}^2$,

$$\phi_1 \circ \pi \circ j \circ \psi_3^{-1}(x, y) = \phi_1 \circ \pi \circ j(x : y : 1) = \phi_1 \circ \pi(x : y : 1) = \phi_1(x + iy : 1) = (x, y),$$

so $\pi \circ j$ is just the identity map in coordinates for points in U_3. Now, $(1 : 0 : 1) \in U_3$, so

$$\pi \circ j(1 : 0 : 1) = \pi(1 : 0 : 1) = (1 : 1)$$

is a regular value of $\pi \circ j$. Therefore,

$$\deg(\pi \circ j) = \#(\pi \circ j)^{-1}(1 : 1) \pmod 2 = 1 \pmod 2,$$

since $(\pi \circ j)^{-1}(1 : 1) = \{(1 : 0 : 1)\}$. Therefore, $\pi \circ j$ has non-zero, odd degree, and so j must also have odd degree. Therefore,

$$j^* : H^2(\mathbb{CP}^2 \setminus \{(1 : i : 0)\}) \to H^2(\mathbb{RP}^2) \tag{1}$$

is non-zero; since $a^* : H^2(\mathbb{CP}^2) \to H^2(\mathbb{CP}^2 \setminus \{(1 : i : 0)\})$ is an isomorphism, where $a : \mathbb{CP}^2 \setminus \{(1 : i : 0)\} \to \mathbb{CP}^2$ is the natural inclusion, (1) implies that $i^* : H^2(\mathbb{CP}^2) \to H^2(\mathbb{RP}^2)$ is non-zero. Since $H^2(\mathbb{RP}^2) \simeq \mathbb{Z}/2$, this implies that i^* is surjective.

To bring it back to $\mathbb{CP}^\infty$ and $\mathbb{RP}^\infty$, note that the generator of $H^2(\mathbb{CP}^\infty)$ is simply the generator of $H^2(\mathbb{CP}^2)$, since $\mathbb{CP}^2$ is simply the 4-skeleton in the CW complex of $\mathbb{CP}^\infty$. Similarly, the generator of $H^1(\mathbb{RP}^\infty)$ is simply the generator of $H^1(\mathbb{RP}^2)$. Therefore, we've shown above that $i : \mathbb{RP}^\infty \to \mathbb{CP}^\infty$ induces non-zero

$$i^* : H^2(\mathbb{CP}^\infty) \to H^2(\mathbb{RP}^\infty).$$

By the argument given at the beginning of this proof, this in turn implies that f^* is surjective in every even dimension. □

4

Show that the two definitions of Hopf invariant for maps $f : S^{2n-1} \to S^n$ that we discussed in class are equivalent.

No.

5

Parametrize a two torus by $\{(z,w) | |z| = |w| = 1\}$. Let M^6 be the homogeneous space $SU(3)/\text{diag}(z, w, \overline{zw})$.

Let the two torus also act on $SU(3)$ by sending

$$A \mapsto \text{diag}(z, w, zw)A\text{diag}(1, 1, \bar{z}^2\bar{w}^2)$$

and let N^6 be the quotient.

One can show that the cohomology groups for both manifolds are

$$H^0 = H^6 = \mathbb{Z}, H^1 = H^5 = 0, H^2 = H^4 = \mathbb{Z} \oplus \mathbb{Z}.$$

The ring structure for M^6 is given by

$$\mathbb{Z}[x, y]/(xy + y^2 + x^2, x^3)$$

and that of N^6 by

$$\mathbb{Z}[x, y]/(19x^2 + 9xy + y^2, x^3)$$

where x and y in each case are a basis in dimension 2.

(a): Identify M with a flag manifold.

Answer: Let F be the set of all flags in $\mathbb{C}^3$ and denote elements of F by $(V_1 : V_2)$, where V_1 is a 1-dimensional space and V_2 is a two-dimensional space (obviously, the only possible 3-dimensional space is $\mathbb{C}^3$, so we omit this from the notation; note that we're talking about complex dimensions here). Let U_3 act on F by

$$A \cdot (V_1 : V_2) = (AV_1 : AV_2).$$

Now, if $(V_1, V_2) \in F$, then $V_1 = \langle v_1 \rangle$ and $V_2 = \langle v_1, v_2 \rangle$ for $v_1, v_2 \in \mathbb{C}^3$. Now, suppose $(V_1 : V_2), (W_1 : W_2) \in F$ with $V_1 = \langle v_1 \rangle$, $V_2 = \langle v_1, v_2 \rangle$, $W_1 = \langle w_1 \rangle$ and $W_2 = \langle w_1, w_2 \rangle$. Then, since $U(3)$ is simply the group of all rigid motions in $\mathbb{C}^3$, there exists an element $A \in U(3)$ such that $A : \langle v_1, v_2 \rangle \mapsto \langle w_1, w_2 \rangle$; hence,

$$A \cdot (V_1 : V_2) = (W_1 : W_2).$$

Since our choice of $(V_1 : V_2)$ and $(W_1 : W_2)$ were arbitrary, we see that $U(3)$ acts transitively on F.

Therefore, since F is equal to the orbit of $(V_1 : V_2)$ for any $(V_1 : V_2) \in F$, $F = U(3)/\text{Stab}(V_1 : V_2)$ for $(V_1 : V_2) \in F$. Now, for such a $(V_1 : V_2)$, $V_1 = \langle v_1 \rangle$, $V_2 = \langle v_1, v_2 \rangle$ and if $A \cdot (V_1 : V_2) = (V_1 : V_2)$, then

$$Av_1 = \lambda_1 v_1, \qquad Av_2 = \lambda_2 v_2$$

and, if $v_3 \notin V_2$ such that $\{v_1, v_2, v_3\}$ gives a basis for $\mathbb{C}^3$, then $Av_3 = \lambda_3 v_3$ for $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{C}^*$. Since the only possible eigenvalues of an element of $U(3)$ have norm one and, for any triple $(\lambda_1, \lambda_2, \lambda_3)$ with $|\lambda_i| = 1$, there exists an element of $U(3)$ with $\lambda_1, \lambda_2, \lambda_3$ as eigenvalues (e.g. $\text{diag}(\lambda_1, \lambda_2, \lambda_3)$), we see that the stabilizer of $(V_1 : V_2)$ is $S^1 \times S^1 \times S^1 = T^3$, represented as a subspace of $U(3)$ by

$$T^3 = \{\text{diag}(\lambda_1, \lambda_2, \lambda_3)\} = \left\{\text{diag}(1, 1, \lambda_1\lambda_2\lambda_3) \cdot \text{diag}(\lambda_1, \lambda_2, \overline{\lambda_1\lambda_2})\right\} = S^1 \times \{\text{diag}(\lambda_1, \lambda_2, \overline{\lambda_1\lambda_2})\}.$$

That is, $F = U(3)/T^3$. Now, we also know that $U(3) = S^1 \times SU(3)$. Hence,

$$F = U(3)/T^3 = (S^1 \times SU(3))/(S^1 \times \{\text{diag}(\lambda_1, \lambda_2, \overline{\lambda_1\lambda_2})\}) = SU(3)/\text{diag}(\lambda_1, \lambda_2, \overline{\lambda_1\lambda_2}) = M^6.$$

Thus, we've identified M with the flag manifold F.

♣

(b): Show the action of T^2 in the definition of N is free.

Proof. Suppose

$$A = \begin{pmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{pmatrix} \in SU(3)$$

and $(z, w) \in T^2$ such that $(z, w) \cdot A = A$. Then

$$\begin{aligned} A = (z, w) \cdot A &= \begin{pmatrix} z & 0 & 0 \\ 0 & w & 0 \\ 0 & 0 & zw \end{pmatrix} \begin{pmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \bar{z}^2\bar{w}^2 \end{pmatrix} \\ &= \begin{pmatrix} za_1 & za_2 & \bar{z}\bar{w}^2 a_3 \\ wa_4 & wa_5 & \bar{z}^2\bar{w}a_6 \\ zwa_7 & zwa_8 & \bar{z}\bar{w}a_9 \end{pmatrix} \end{aligned}$$

Then either a_1, a_2 or a_3 is non-zero; if a_1 or a_2 is non-zero, then $z = 1$. In turn, since one of a_7, a_8, a_9 is non-zero, this, coupled with the fact that $z = 1$ implies that $w = 1$. Hence, $(z, w) = 1$. On the other hand, if $a_3 \neq 0$, then $\bar{z}\bar{w}^2 = 1$; hence, $\bar{z} = w^2$. Since either a_7, a_8 or a_9 is non-zero, this implies that $1 = zw = \bar{w}^2 w = \bar{w}$, so $w = z = 1$, or $1 = \bar{z}\bar{w} = w^2\bar{w} = w$, so $w = z = 1$. Therefore, we see that $(z, w) \cdot A = A$ implies that $(z, w) = (1, 1)$. Since our choice of A was arbitrary, we see that this holds for all $A \in SU(3)$, so the action of T^2 on $SU(3)$ is free. □

(c): Are M and N homotopy equivalent?

Answer: Suppose there exists a homotopy $f : M \to N$. Then

$$f^* : H^2(N) \to H^2(M)$$

must be an isomorphism. Then, since $H^*(N) = \mathbb{Z}[\alpha, \beta]/(19\alpha^2 + 9\alpha\beta + \beta^2, \alpha^3)$ where α, β a basis in $H^2(N)$ and $H^*(M) = \mathbb{Z}[x, y]/(x^2 + xy + y^2, x^3)$ where x, y a basis in $H^2(M)$, we must have

$$\alpha \mapsto ax + by, \qquad \beta \mapsto cx + dy$$

where

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}).$$

Now

$$0 = f^*(0) = f^*(\alpha^3) = (f^*(\alpha))^3 = (ax + by)^3.$$

Also,

$$0 = f^*(0) = f^*(19\alpha^2 + 9\alpha\beta + \beta^2) = 19(ax+by)^2 + 9(ax+by)(cx+dy) + (cx+dy)^2.$$

Now, modulo 3, these reduce to

$$a^3x^3 + b^3y^3 = 0 \tag{2}$$

and

$$(ax + by)^2 + (cx + dy)^2 = 0. \tag{3}$$

Since $x^3 = 0$, this implies that $b^3y^3 = 0$; since $y^3 \neq 0$, this implies $b = 0$. Hence, since $1 = \det A = ad - bc = ad$, we have two cases:

Case I: $a = d = 1$. Then (3) reduces to

$$0 = x^2 + c^2x^2 + 2cxy + y^2 = (-xy - y^2) + c^2x^2 + 2cxy + y^2 = c^2x^2 + (2c-1)xy.$$

Hence, $c^2x^2 = (1 - 2c)xy$, or $c^2x = (1 - 2c)y$, which is impossible since x and y are linearly independent.

Case II: $a = d = -1$. Then (**??**) reduces to

$$0 = x^2 + c^2x^2 - 2cxy + y^2 = (-xy - y^2) + c^2x^2 - 2cxy + y^2 = c^2x^2 - (2c+1)xy.$$

Hence, $c^2x^2 = (2c + 1)xy$, or $c^2x = (2c + 1)y$, which is again impossible since x and y are linearly independent.

Therefore, we see that there is no such homotopy $f : M \to N$.

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