GEOMETRY HW 2

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1

Let G be a connected Lie group. Show that the fundamental group of G is abelian.

2

Relate the fundamental group of two n-dimensional manifolds M and N to that of the connected sum $M\#N$.

3

Compute the fundamental group of an orientable surface M_g of genus g and show that M_g is not homotopy equivalent to M_h if $g \neq h$.

4

Show that a local homeomorphism $f : M \to B$ is a covering if M is compact Hausdorff.

5

Let $f : X \to Z$, $g : X \to Y$ and $h : Y \to Z$ be continuous maps such that $f = h \circ g$. Show that h is a covering if f and g are coverings.

Proof. Suppose f and g are coverings. Let $p \in Z$. Then, since f is a covering, there exists a neighborhood U of p such that U is evenly covered by f. Specifically,

$$f^{-1}(U) = \bigsqcup_i U_i$$

for disjoint open sets $U_i \subset X$ such that $f : U_i \to U$ is a homeomorphism for all i. Let $x \in f^{-1}(p)$ and let $y = g(x)$. Note that $p = f(x) = h \circ g(x) = h(y)$. Since g is a covering, there exists a neighborhood V of y such that V is evenly covered by g. Namely,

$$g^{-1}(V) = \bigsqcup_j V_j$$

for disjoint open sets $V_j \subset X$ such that $f : V_j \to V$ is a homeomorphism for all j. □

6

If M is a connected, non-orientable manifold, show that there exists a connected twofold cover of M which is orientable.

7

Show that the Klein bottle has a twofold cover which is orientable (describe it explicitly), and another one which is non-orientable.

8

Describe explicitly all possible coverings of an annulus $\{x \in \mathbb{R}^2 | 1 < |x| < 4\}$.

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