GEOMETRY HW 4

CLAY SHONKWILER

1

Show that a connected Lie group acting on a topological space X induces trivial maps in homology. Is this also true if G is a finite group?

2

Let G be a connected, simply connected Lie group and Γ a discrete subgroup of the center of G. Show that $G \to G/\Gamma$ is a covering. Show that every connected Lie group arises from this construction. Find all connected Lie groups whose universal cover is $S^3 \times S^3$.

Proof. As we saw last semester, Γ acts freely and properly discontinuously on G and G/Γ is a Lie group. Also, as we saw on Homework 3, Problem 2, $G \to G/\Gamma$ is a covering.

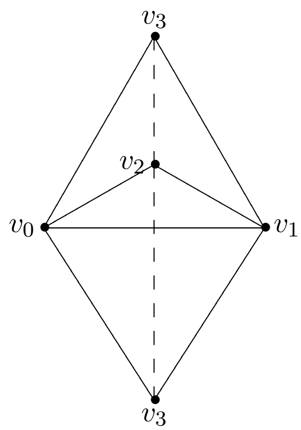
Now, suppose G is a connected Lie group, and let E be the universal cover of G. Then, as we saw on Homework 3, Problem 4, E is a Lie group and $\sigma : E \to G$ is a homomorphism. Let D be the deck group of $\sigma : E \to G$; then $G = E/D$. Let $\widetilde{e} \in E$ be the identity element, with $\sigma(\widetilde{e}) = e$, where e is the identity in G. For any $\phi \in D$, let $e_\phi = \phi(e)$ and consider the set $H = \{e_\phi | \phi \in D\}$. Then $H \subseteq \sigma^{-1}(e)$, and we claim that H is a discrete subgroup of the center of G isomorphic to D. Note that if $\phi_1(e) = \phi_2(e)$, then $\phi_1 = \phi_2$, so there is a bijection between H and D. Now, suppose e_ϕ, $e_\psi \in H$. Then

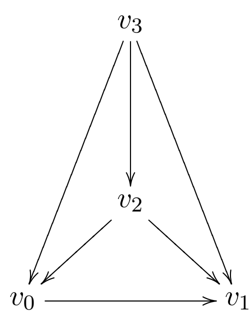
$$e_\phi \cdot e_\psi = \phi(e) \cdot \psi(e) =$$

□

3

Determine the simplicial homology of S^3 from an appropriate Δ complex.





4

Compute the simplicial homology of the Δ complex obtained from one simplex by identifying all faces of the same dimension.

5

Show that the inclusion $A \to X$ induces an isomorphism in homology iff $H_*(X, A) = 0$.

6

Show that $\widetilde{H}_n(X)$ is isomorphic to $\widetilde{H}_{n+1}(SX)$ where SX is the suspension of X.

7

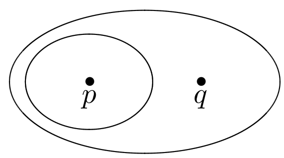
Find an example where $H_*(X, A)$ is different from $H_*(X/A)$.

Example: Let $X = \mathbb{R}^n$ and let $A = \mathbb{R}^n - \{0\}$. Then S^{n-1} is a deformation retract of $\mathbb{R}^n - \{0\}$ (as we've seen, e.g., on Homework 1), so

$$\widetilde{H}_i(\mathbb{R}^n, \mathbb{R}^n - \{0\}) = \widetilde{H}_i(\mathbb{R}^n, S^{n-1}) = \widetilde{H}_i(D^n, \partial D^n) = \begin{cases} \mathbb{Z} & \text{if } i = n \\ 0 & \text{otherwise,} \end{cases}$$

where the second equality arises because D^n is a deformation retract of $\mathbb{R}^n$ and $S^{n-1} = \partial D^n$.

On the other hand, X/A is a two point set with one open and one closed point; i.e., the open sets are as pictured below:



Now, define $F : I \times X/A \to X/A$ by

$$F(1, p) = F(1, q) = q$$
$$F(t, q) = q \quad \forall t$$
$$F(t, p) = p \quad \forall t < 1$$

Then $F^{-1}(\{p\} = [0, 1) \times \{p\}$, which is open, since $[0, 1) \subset [0, 1]$ is open and $\{p\} \subset X/A$ is open, and $F^{-1}(\{p, q\}) = I \times X/A$, which is open. Thus, F is continuous, so we see that X/A is contractible. Therefore,

$$\widetilde{H}_i(X/A) = 0$$

for all i. Hence, $H_n(X, A) \not\cong H_n(X/A)$.

$\clubsuit$

8

Show that a map $f : X \to Y$ is a covering if it has the unique homotopy lifting property. Find an example which has the unique path lifting property but not the unique homotopy lifting property.

DRL 3E3A, University of Pennsylvania
E-mail address: shonkwil@math.upenn.edu