

GEOMETRY HW 5

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1

Give an example that excision does not hold if we only assume $A \subset B$ instead of $\bar{A} \subset \overset{\circ}{B}$.

Example: Let $X = \mathbb{R}^n$ for $n > 2$ and let $A = B = \mathbb{R}^n - \{0\}$. Then $A \subset B$. However, as we showed on last week's homework,

$$H_n(X, B) = H_n(\mathbb{R}^n, \mathbb{R}^n - \{0\}) = \mathbb{Z},$$

whereas $\mathbb{R}^n - (\mathbb{R}^n - \{0\}) = 0$, so

$$H_n(X - A, B - A) = H_n(\mathbb{R}^n - (\mathbb{R}^n - \{0\}), \emptyset) = H_n(\{0\}, \emptyset) = H_n(\{0\}) = 0$$

since $n > 2 > 0$.



2

Assume that X is a compact, connected topological space with $\pi_1(X)$ equal to the symmetric group S_3 , $H_2(X, \mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z}/6$, and $H_3(X, \mathbb{Z}) = \mathbb{Z}^2 \oplus \mathbb{Z}/7$. Compute $H_*(X, \mathbb{Z}/2)$ and $H_*(X, \mathbb{Z}/14)$.

Answer: Throughout, we will use the short exact sequences:

$$(1) \quad 0 \rightarrow H_i(X, \mathbb{Z}) \otimes G \rightarrow H_i(X, G) \rightarrow \text{Tor}(H_{i-1}(X, \mathbb{Z}), G) \rightarrow 0$$

to compute $H_i(X, G)$. First, when $G = \mathbb{Z}/2$, we know that $H_0(X, \mathbb{Z}) = \mathbb{Z}$ since X is connected, and $H_{-1}(X, \mathbb{Z}) = 0$. Also, $\mathbb{Z} \otimes \mathbb{Z}/2 = \mathbb{Z}/2$. Hence, for $i = 0$, (1) reduces to

$$0 \rightarrow \mathbb{Z} \otimes \mathbb{Z}/2 \rightarrow H_0(X, \mathbb{Z}/2) \rightarrow 0 \rightarrow 0;$$

since this sequence is exact and $\mathbb{Z} \otimes \mathbb{Z}/2 = \mathbb{Z}/2$, we see that $H_0(X, \mathbb{Z}/2) = \mathbb{Z}/2$.

Now, for $i = 1$, $H_1(X, \mathbb{Z}) = \pi_1(X)/[\pi_1, \pi_1]$; since $\pi_1(X) = S_3 \langle a, b \mid a^3 = b^2 = 1, ab = ba^2 \rangle$ and the commutator of this group is $\langle a \mid a^3 = 1 \rangle \simeq \mathbb{Z}/3$, we see that $H_1(X, \mathbb{Z}) \simeq X_3/\mathbb{Z}/3 \simeq \mathbb{Z}/2$. Now, since X connected, $H_0(X, \mathbb{Z}) = \mathbb{Z}$. Furthermore,

$$(2) \quad 0 \rightarrow \mathbb{Z} \xrightarrow{-2} \mathbb{Z} \rightarrow \mathbb{Z}/2 \rightarrow 0$$

is a free resolution of $\mathbb{Z}/2$, so applying $\mathbb{Z} \otimes \cdot$ and cutting out the last term yields:

$$0 \rightarrow \mathbb{Z} \otimes \mathbb{Z} \xrightarrow{\partial_1} \mathbb{Z} \otimes \mathbb{Z} \xrightarrow{\partial_0} 0,$$

where $\partial_1 = 1 \otimes (\cdot 2)$. Also, $\mathbb{Z} \otimes \mathbb{Z} = \mathbb{Z}$, so this further reduces to

$$0 \rightarrow \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \rightarrow 0;$$

$\text{Tor}(\mathbb{Z}, \mathbb{Z}/2) = \ker \partial_1 = 0$. Hence, for $i = 1$, (1) reduces to

$$0 \rightarrow \mathbb{Z}/2 \otimes \mathbb{Z}/2 \rightarrow H_1(X, \mathbb{Z}/2) \rightarrow 0 \rightarrow 0;$$

since this sequence is exact and $\mathbb{Z}/2 \otimes \mathbb{Z}/2 = \mathbb{Z}/2$, this means that $H_1(X, \mathbb{Z}/2) = \mathbb{Z}/2$.

For $i = 2$, $H_2(X, \mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z}/6$. We know $H_1(X, \mathbb{Z}) = \mathbb{Z}/2$ from above; applying $\mathbb{Z}/2 \otimes \cdot$ to the free resolution (2) and chopping the last term yields:

$$0 \rightarrow \mathbb{Z}/2 \otimes \mathbb{Z} \xrightarrow{\partial_1} \mathbb{Z}/2 \otimes \mathbb{Z} \xrightarrow{\partial_0} 0$$

where $\partial_1 = 1 \otimes \cdot 2$. Since $\mathbb{Z}/2 \otimes \mathbb{Z} = \mathbb{Z}/2$, reducing ∂_1 to simply $\cdot 2$, we see that $\text{Tor}(\mathbb{Z}/2, \mathbb{Z}/2) = \ker \partial_1 = \mathbb{Z}/2$. Hence, for $i = 2$, (1) reduces to

$$0 \rightarrow (\mathbb{Z} \oplus \mathbb{Z}/6) \otimes \mathbb{Z}/2 \rightarrow H_2(X, \mathbb{Z}/2) \rightarrow \mathbb{Z}/2 \rightarrow 0.$$

Now,

$$(\mathbb{Z} \oplus \mathbb{Z}/6) \otimes \mathbb{Z}/2 = (\mathbb{Z} \otimes \mathbb{Z}/2) \oplus (\mathbb{Z}/6 \otimes \mathbb{Z}/2) = \mathbb{Z}/2 \oplus \mathbb{Z}/2;$$

since the sequence splits, $H_2(X, \mathbb{Z}/2) = (\mathbb{Z}/2 \oplus \mathbb{Z}/2) \oplus \mathbb{Z}/2 = (\mathbb{Z}/2)^3$.

For $i = 3$, $H_3(X, \mathbb{Z}) = \mathbb{Z}^2 \oplus \mathbb{Z}/7$. We know $H_2(X, \mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z}/6$ from above; applying $(\mathbb{Z} \oplus \mathbb{Z}/6) \otimes \cdot$ to the free resolution (2) and chopping the last term yields:

$$0 \rightarrow (\mathbb{Z} \oplus \mathbb{Z}/6) \otimes \mathbb{Z} \xrightarrow{\partial_1} (\mathbb{Z} \oplus \mathbb{Z}/6) \otimes \mathbb{Z} \rightarrow 0$$

where $\partial_1 = 1 \otimes \cdot 2$. Since

$$(\mathbb{Z} \oplus \mathbb{Z}/6) \otimes \mathbb{Z} = (\mathbb{Z} \otimes \mathbb{Z}) \oplus (\mathbb{Z}/6 \otimes \mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z}/6,$$

we see that $\text{Tor}(\mathbb{Z} \oplus \mathbb{Z}/6, \mathbb{Z}/2) = 0 \oplus (\mathbb{Z}/6)/(\mathbb{Z}/3) \simeq 0 \oplus \mathbb{Z}/2 \simeq \mathbb{Z}/2$. Hence, for $i = 3$, (1) reduces to

$$0 \rightarrow (\mathbb{Z}^2 \oplus \mathbb{Z}/7) \otimes \mathbb{Z}/2 \rightarrow H_3(X, \mathbb{Z}/2) \rightarrow \mathbb{Z}/2 \rightarrow 0.$$

Since

$$(\mathbb{Z}^2 \oplus \mathbb{Z}/7) \otimes \mathbb{Z}/2 = (\mathbb{Z}^2 \otimes \mathbb{Z}/2) \oplus (\mathbb{Z}/7 \otimes \mathbb{Z}/2) = ((\mathbb{Z} \otimes \mathbb{Z}/2) \oplus (\mathbb{Z} \otimes \mathbb{Z}/2)) \oplus 0 = \mathbb{Z}/2 \oplus \mathbb{Z}/2$$

and the sequence splits, this implies that $H_3(X, \mathbb{Z}/2) = (\mathbb{Z}/2)^3$.

Now we turn our attention to $G = \mathbb{Z}/14$. In the zero case, $H_0(X, \mathbb{Z}) = \mathbb{Z}$ and $H_{-1}(X, \mathbb{Z}) = 0$, so $\text{Tor}(H_{-1}(X, \mathbb{Z}), \mathbb{Z}/14) = \text{Tor}(0, \mathbb{Z}/14) = 0$ and, thus, (1) reduces to

$$0 \rightarrow \mathbb{Z} \otimes \mathbb{Z}/14 \rightarrow H_0(X, \mathbb{Z}/14) \rightarrow 0 \rightarrow 0;$$

since this sequence is exact and $\mathbb{Z} \otimes \mathbb{Z}/14 = \mathbb{Z}/14$, this implies that $H_0(X, \mathbb{Z}/14) = \mathbb{Z}/14$.

For $i = 1$, we know, from above, that $H_1(X, \mathbb{Z}) = \mathbb{Z}/2$ and, obviously, $H_0(X, \mathbb{Z}) = \mathbb{Z}$. Now, the free resolution of $\mathbb{Z}/14$ is

$$(3) \quad 0 \rightarrow \mathbb{Z} \xrightarrow{\cdot 14} \mathbb{Z} \rightarrow \mathbb{Z}/14 \rightarrow 0.$$

Hence, applying $\mathbb{Z} \otimes \cdot$ to this free resolution and chopping the last term yields

$$0 \rightarrow \mathbb{Z} \otimes \mathbb{Z} \xrightarrow{\partial_1} \mathbb{Z} \otimes \mathbb{Z} \rightarrow 0$$

where $\partial_1 = 1 \otimes \cdot 14$. Since $\mathbb{Z} \otimes \mathbb{Z} = \mathbb{Z}$ and the only element of \mathbb{Z} that maps to 0 under multiplication by 14 is 0, we see that $\text{Tor}(\mathbb{Z}, \mathbb{Z}/14) = \ker \partial_1 = 0$. Hence, for $i = 1$, (1) reduces to

$$0 \rightarrow \mathbb{Z}/2 \otimes \mathbb{Z}/14 \rightarrow H_1(X, \mathbb{Z}/14) \rightarrow 0 \rightarrow 0.$$

Since $\mathbb{Z}/2 \otimes \mathbb{Z}/14 = \mathbb{Z}/2$, we see that $H_1(X, \mathbb{Z}/14) = \mathbb{Z}/2$.

For $i = 2$, we know that $H_2(X, \mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z}/6$ and that $H_1(X, \mathbb{Z}) = \mathbb{Z}/2$. Hence, applying $\mathbb{Z}/2 \otimes \cdot$ to the free resolution (3) and cutting off the trailing term yields

$$0 \rightarrow \mathbb{Z}/2 \otimes \mathbb{Z} \xrightarrow{\partial_1} \mathbb{Z}/2 \otimes \mathbb{Z} \rightarrow 0$$

where $\partial_1 = 1 \otimes \cdot 14$. Since $\mathbb{Z}/2 \otimes \mathbb{Z} = \mathbb{Z}/2$, we see that $\text{Tor}(\mathbb{Z}/2, \mathbb{Z}/14) = \ker \partial_1 = \mathbb{Z}/2$. Thus, for $i = 2$, (1) reduces to

$$0 \rightarrow (\mathbb{Z} \oplus \mathbb{Z}/6) \otimes \mathbb{Z}/14 \rightarrow H_2(X, \mathbb{Z}/14) \rightarrow \mathbb{Z}/2 \rightarrow 0.$$

Now,

$$(\mathbb{Z} \oplus \mathbb{Z}/6) \otimes \mathbb{Z}/14 = (\mathbb{Z} \otimes \mathbb{Z}/14) \oplus (\mathbb{Z}/6 \otimes \mathbb{Z}/14) = \mathbb{Z}/14 \oplus \mathbb{Z}/6,$$

so, since the above sequence splits, we see that

$$H_2(X, \mathbb{Z}/14) = \mathbb{Z}/14 \oplus \mathbb{Z}/6 \oplus \mathbb{Z}/2 = \mathbb{Z}/42 \oplus (\mathbb{Z}/2)^2.$$

Finally, for $i = 3$, we know that $H_3(X, \mathbb{Z}) = \mathbb{Z}^2 \oplus \mathbb{Z}/7$ and $H_2(X, \mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z}/6$. Hence, applying $(\mathbb{Z} \oplus \mathbb{Z}/6) \otimes \cdot$ to the free resolution (3) and cutting off the last term yields

$$0 \rightarrow (\mathbb{Z} \oplus \mathbb{Z}/6) \otimes \mathbb{Z} \xrightarrow{\partial_1} (\mathbb{Z} \oplus \mathbb{Z}/6) \otimes \mathbb{Z} \rightarrow 0$$

where $\partial_1 = 1 \otimes \cdot 14$. Since $(\mathbb{Z} \oplus \mathbb{Z}/6) \otimes \mathbb{Z} = \mathbb{Z} \oplus \mathbb{Z}/6$, we reduce ∂_1 to multiplication by 14 and so, since the only non-zero element of $\mathbb{Z}/6$ which, when multiplied by 14, is congruent to 0 (mod 6) is 3, we see that $\text{Tor}(\mathbb{Z} \oplus \mathbb{Z}/6, \mathbb{Z}/14) = \ker \partial_1 = 0 \oplus \mathbb{Z}/2 \simeq \mathbb{Z}/2$. Thus, when $i = 3$, (1) reduces to

$$0 \rightarrow (\mathbb{Z}^2 \oplus \mathbb{Z}/7) \otimes \mathbb{Z}/14 \rightarrow H_3(X, \mathbb{Z}/14) \rightarrow \mathbb{Z}/2 \rightarrow 0.$$

Now,

$$(\mathbb{Z}^2 \oplus \mathbb{Z}/7) \otimes \mathbb{Z}/14 = (\mathbb{Z} \otimes \mathbb{Z}/14) \oplus (\mathbb{Z} \otimes \mathbb{Z}/14) \oplus (\mathbb{Z}/7 \otimes \mathbb{Z}/14) = \mathbb{Z}/14 \oplus \mathbb{Z}/14 \oplus \mathbb{Z}/7$$

and so, since the sequence splits, we see that

$$H_3(X, \mathbb{Z}/14) = \mathbb{Z}/14 \oplus \mathbb{Z}/14 \oplus \mathbb{Z}/7 \oplus \mathbb{Z}/2 = (\mathbb{Z}/14)^3.$$



3

Let X be a topological manifold. Relate, as much as you can, the homology of $X - p$ for $p \in X$ with the homology of X . In case any one wonders, you can assume X is connected.

Answer: Note that, since X is connected, if $n > 1$ and X is an n -dimensional topological manifold, then $X - \{p\}$ is connected, so $H_0(X - \{p\}) = \mathbb{Z} = H_0(X)$.

If $n = 1$, then there are only 2 1-dimensional manifolds up to homeomorphism, \mathbb{R} and S^1 . If $X = \mathbb{R}$, then $\mathbb{R} - \{p\}$ is disconnected, with each piece homeomorphic to \mathbb{R} . So each piece will have $H_0 = \mathbb{Z}$ and $H_1 = 0$. If $X = S^1$, then $X - \{p\}$ is connected and homeomorphic to \mathbb{R} , so $H_0(X - \{p\}) = \mathbb{Z} = H_0(X)$ and $H_1(X - \{p\}) = 0$.

In general, we will use the Mayer Vietoris sequence:

$$\cdots \rightarrow H_i(U \cap V) \rightarrow H_i(U) \oplus H_i(V) \rightarrow H_i(U \cup V) \rightarrow H_{i-1}(U \cap V)$$

with U an open ball homeomorphic to D^n about p and $V = X - \{p\}$. Then $U \cup V = X$ and $U \cap V = U - \{p\}$. Note that S^{n-1} is a deformation retract of $D^n - \{0\}$, so S^{n-1} is homeomorphic to a deformation retract of $U - \{p\} = U \cap V$. For $i > 0$, $H_i(U) = 0$ since U is contractible. Hence, the Mayer-Vietoris sequence gives us, for $i > 0$,

$$\cdots \rightarrow H_i(S^{n-1}) \rightarrow 0 \oplus H_i(X - \{p\}) \rightarrow H_i(X) \rightarrow H_{i-1}(S^{n-1}) \rightarrow \cdots$$

Since $H_i(S^{n-1}) = 0$ for $0 < i < n - 1$, we see that, for $1 < i < n - 1$, the sequence further reduces to

$$\cdots \rightarrow 0 \rightarrow H_i(X - \{p\}) \rightarrow H_i(X) \rightarrow 0 \rightarrow \cdots,$$

so $H_i(X - \{p\}) \simeq H_i(X)$.

If $n > 2$ and $i = 1$, then the Mayer-Vietoris sequence reduces to

$$0 \rightarrow 0 \oplus H_1(X - \{p\}) \rightarrow H_1(X) \rightarrow \mathbb{Z} \rightarrow \mathbb{Z} \oplus \mathbb{Z} \rightarrow \mathbb{Z} \rightarrow 0,$$

since $H_1(S^{n-1}) = 0$ and $H_0 = \mathbb{Z}$ for any connected space. Hence, since the kernel of $\mathbb{Z} \rightarrow 0$ is \mathbb{Z} , the image of $\mathbb{Z} \oplus \mathbb{Z} \rightarrow \mathbb{Z}$ is \mathbb{Z} , and so the kernel of this map (and, hence, the image of $\mathbb{Z} \rightarrow \mathbb{Z} \oplus \mathbb{Z}$) is also \mathbb{Z} . Since the image of $\mathbb{Z} \rightarrow \mathbb{Z} \oplus \mathbb{Z}$ is \mathbb{Z} , the kernel of this map must be 0. Hence, we can re-write just the H_1 piece as

$$0 \rightarrow 0 \oplus H_1(X - \{p\}) \rightarrow H_1(X) \rightarrow 0,$$

so $H_1(X - \{p\}) \simeq H_1(X)$. This, combined with our results proved above, allows us to conclude that, for any n , $H_i(X - \{p\}) \simeq H_i(X)$ for all $0 \leq i < n - 1$.

Now, consider some examples. If $X = \mathbb{R}^n$, the open ball, then X is certainly a topological manifold and $X - \{p\}$ deformation retracts to S^{n-1} . Hence,

$$H_n(X - \{p\}) = H_n(S^{n-1}) = 0 = H_n(X)$$

and

$$H_{n-1}(X - \{p\}) = H_{n-1}(S^{n-1}) = \mathbb{Z} \neq 0 = H_{n-1}(X).$$

On the other hand, if $X = S^n$, then $S^n - \{p\}$ is contractible and so has trivial homology; hence

$$H_n(X - \{p\}) = 0 \neq \mathbb{Z} = H_n(X)$$

and

$$H_{n-1}(X - \{p\}) = 0 = H_{n-1}(X).$$

Thus, we can't say much of anything about the n th and $(n - 1)$ st homology of $X - \{p\}$ in terms of the homology of X , other than that $H_n(X - \{p\})$ injects into $H_n(X)$.



4

Let $M \# N$ be the connected sum of two n -dimensional manifolds M and N . Relate as much of the homology of $M \# N$ as you can, to the homology of M and N .

Answer: Note that $H_0(M \# N) = \mathbb{Z} = H_0(M) = H_0(N)$. If $n = 1$, then we already saw on homework 2 that the only possibilities are

$$M \# N \approx S^1 \text{ if } M = N = S^1$$

$$M \# N \approx \mathbb{R} \text{ if } M = \mathbb{R}, N = S^1 \text{ (or vice versa)}$$

$$M \# N \approx \mathbb{R} \sqcup \mathbb{R} \text{ if } M = N = \mathbb{R}.$$

In the first case, $H_1(M \# N) = \mathbb{Z} = H_1(M) = H_1(N)$; in the second case, $H_1(M \# N) = 0 = H_1(M)$; and in the third case, $H_1(M \# N) = 0 = H_1(M) = H_1(N)$.

For the general case, we again use the Mayer-Vietoris sequence:

$$\cdots \rightarrow H_i(U \cap V) \rightarrow H_i(U) \oplus H_i(V) \rightarrow H_i(U \cup V) \rightarrow H_{i-1}(U \cap V) \rightarrow \cdots$$

where $U \approx M$ (U is a M plus a tiny strip at the point of attachment of the connected sum), $V \approx N$ and $U \cap V$ can be deformation retracted to S^{n-1} . Then for $0 < i < n - 1$, we know that $H_i(S^{n-1}) = 0$, U is homotopy equivalent to $M - \{p\}$ and V is homotopy equivalent to $N - \{p\}$. Hence, applying the above result that, for $0 < i < n - 1$, $H_i(M - \{p\}) = H_i(M)$ for any manifold M and using the fact that for these i $H_i(U \cap V) = H_i(S^{n-1}) = 0$, we see that the Mayer Vietoris sequence reduces to

$$\cdots \rightarrow 0 \rightarrow H_i(M) \oplus H_i(N) \rightarrow H_i(M \# N) \rightarrow 0 \rightarrow \cdots,$$

so for these i , $H_i(M \# N) = H_i(M) \oplus H_i(N)$. Again, we're not, in general, going to be able to say very much about H_n and H_{n-1} .



For DeRham cohomology you should give a complete answer to problem 3 and 4.

Answer: We know that $H^0(M) = \mathbb{R}$ for any connected manifold M , so $H^0(M^n - \{p\}) = \mathbb{R} = H^0(M^n)$ for $n > 1$. For 1-dimensional manifolds, we only have 2 possibilities, \mathbb{R} and S^1 . $\mathbb{R} - \{p\}$ is disconnected, with each piece diffeomorphic to \mathbb{R}^n . $S^1 - \{p\}$ is diffeomorphic to \mathbb{R} , so $H^0(S^1 - \{p\}) = \mathbb{R} = H^0(S^1)$, but $H^1(S^1 - \{p\}) = 0 \neq \mathbb{R} = H^1(S^1)$.

In general, we will use the Mayer-Vietoris sequence for DeRham cohomology, which is:

$$(4) \quad \cdots \rightarrow H^{i-1}(U \cap V) \rightarrow H^i(U \cup V) \rightarrow H^i(U) \oplus H^i(V) \rightarrow H^i(U \cap V) \rightarrow \cdots,$$

where we'll let U be a small open ball about the point p and $V = M^n - \{p\}$. Then $U \cup V = M$ and $U \cap V = U - \{p\}$, which is homotopic to S^{n-1} . Then, for $1 < i < n-1$, $H^i(S^{n-1}) = 0$ and $H^i(U) = 0$, so the sequence (4) reduces to

$$\cdots \rightarrow 0 \rightarrow H^i(M) \rightarrow 0 \oplus H^i(M - \{p\}) \rightarrow 0,$$

so $H^i(M) \simeq H^i(M - \{p\})$.

Now, we know that $H^n(M) = \mathbb{R}$ if M is compact and orientable and $H^n(M) = 0$ otherwise. Deleting a point from a compact space makes it non-compact, and deleting a point from a non-compact (or non-orientable) manifold cannot make the manifold compact (or orientable), so we see that $H^n(M - \{p\}) = 0$ regardless of what M is. Therefore, if M is compact and orientable, then the end of the Mayer-Vietoris sequence becomes (since $H^{n-1}(S^{n-1}) = \mathbb{R}$):

$$0 \rightarrow H^{n-1}(M) \rightarrow 0 \oplus H^{n-1}(M - \{p\}) \rightarrow \mathbb{R} \rightarrow \mathbb{R} \rightarrow 0.$$

Since $\mathbb{R} \rightarrow \mathbb{R} \rightarrow 0$ is exact, we see that the kernel of $\mathbb{R} \rightarrow \mathbb{R}$ is 0, so we see that $H^{n-1}(M) \simeq H^{n-1}(M - \{p\})$.

On the other hand, if M is non-compact or non-orientable or both, then (4) reduces to

$$0 \rightarrow H^{n-1}(M) \rightarrow 0 \oplus H^{n-1}(M - \{p\}) \rightarrow \mathbb{R} \rightarrow 0,$$

so $\mathbb{R} \oplus H^{n-1}(M) \simeq H^{n-1}(M - \{p\})$ since the sequence is split.

Therefore, we conclude that $H^i(M - \{p\}) \simeq H^i(M)$ for $0 \leq i \leq n-1$, that $H^n(M - \{p\}) = 0$ and that the $n-1$ case is as described above.

Turning to $M \# N$, we will again use the Mayer-Vietoris sequence for DeRham cohomology, but now we will let U and V be as in problem 4 above; namely U is M together with a tiny strip near the removed disc, V is N together with a tiny strip near the removed disc, and $U \cap V$ is a small cylinder in this region. Then, as before, U is homotopy equivalent to $M - \{p\}$, V is homotopy equivalent to $N - \{q\}$, and $U \cap V$ is homotopy equivalent to S^{n-1} . In the zero case, $H^0(M) = \mathbb{R}$ for any manifold M , so

$H^0(M\#N) = \mathbb{R} = H^0(M) = H^0(N)$. Now, for $0 < i < n - 1$, (4) reduces to

$$\cdots \rightarrow 0 \rightarrow H^i(M\#N) \rightarrow H^i(M - \{p\}) \oplus H^i(N - \{q\}) \rightarrow 0 \rightarrow \cdots$$

since $H^i(M - \{p\}) = H^i(M)$ and $H^i(N - \{q\}) = H^i(N)$ (as we just showed), we see that, for $0 < i < n - 1$,

$$H^i(M\#N) = H^i(M) \oplus H^i(N).$$

In the top dimension, there are a few different cases we have to worry about. If M and N are both compact and orientable, then $M\#N$ is compact and orientable, so $H^n(M\#N) = \mathbb{R} = H^n(M) = H^n(N)$. Also, we know from above that $H^{n-1}(M - \{p\}) = H^{n-1}(M)$ (and similarly for N). Hence, at the $n - 1$ spot, the Mayer-Vietoris sequence becomes:

$$\cdots \rightarrow 0 \rightarrow H^{n-1}(M\#N) \rightarrow H^{n-1}(M) \oplus H^{n-1}(N) \rightarrow \mathbb{R} \rightarrow \mathbb{R} \rightarrow 0.$$

Since the kernel of $\mathbb{R} \rightarrow 0$ must be \mathbb{R} , this is the image of $\mathbb{R} \rightarrow \mathbb{R}$, so the kernel of this map is 0. Hence, we can reduce the above to

$$\cdots \rightarrow 0 \rightarrow H^{n-1}(M\#N) \rightarrow H^{n-1}(M) \oplus H^{n-1}(N) \rightarrow 0.$$

Therefore, $H^{n-1}(M\#N) = H^{n-1}(M) \oplus H^{n-1}(N)$.

If neither M nor N is compact and orientable, then we claim that $M\#N$ cannot be compact and orientable. If either M or N is not orientable then suppose, without loss of generality, that N is not orientable. If $M\#N$ is orientable, then this orientation remains an orientation if we cut M out of $M\#N$. What we're left with after doing so is homotopy equivalent to N minus a point, which is not orientable. Therefore $M\#N$ is not orientable. On the other hand, if M and N are orientable but non-compact, let $\{U_\alpha\}$ be an open cover N . Then this can be extended to an open cover of $M\#N$; if $M\#N$ were compact, then there would exist a finite subcover; which, when restricted to N , would be a finite subcover of N minus a point. Since N is not compact, neither is N minus a point, so we conclude that $M\#N$ is not compact. Therefore, in this instance, $H^n(M\#N) = 0 = H^n(M) = H^n(N)$. For $i = n - 1$, we saw that $H^{n-1}(M - \{p\}) = \mathbb{R} \oplus H^{n-1}(M)$ (and similarly for N), so the Mayer-Vietoris sequence becomes:

$$\cdots \rightarrow 0 \rightarrow H^{n-1}(M\#N) \rightarrow (\mathbb{R} \oplus H^{n-1}(M)) \oplus (\mathbb{R} \oplus H^{n-1}(N)) \rightarrow \mathbb{R} \rightarrow 0.$$

Since this sequence is split, we see that

$$H^{n-1}(M\#N) = \mathbb{R} \oplus H^{n-1}(M) \oplus H^{n-1}(N).$$

Finally, suppose M is compact and orientable, but N is not. Then the argument given above also demonstrates that $M\#N$ cannot be compact and orientable, so $H^n(M\#N) = 0 = H^n(N)$, but $H^n(M) = \mathbb{R}$. In the $n - 1$ case, we know that $H^{n-1}(M - \{p\}) = H^{n-1}(M)$ and $H^{n-1}(N - \{q\}) = \mathbb{R} \oplus H^{n-1}(N)$, so the Mayer-Vietoris sequence reduces to

$$\cdots \rightarrow 0 \rightarrow H^{n-1}(M\#N) \rightarrow H^{n-1}(M) \oplus H^{n-1}(N) \oplus \mathbb{R} \rightarrow \mathbb{R} \rightarrow 0.$$

Hence, $H^{n-1}(M\#N) = H^{n-1}(M) \oplus H^{n-1}(N)$ since the sequence splits.



6

Relate the Euler characteristic of $M\#N$ with that of M and N . You may want to use DeRham cohomology. No assumptions on M and N .

Answer: We know that the Euler characteristic

$$\chi(M) = \sum_{i=0}^n (-1)^i b_i$$

where $b_i = \dim H_{DR}^i(M)$ for any manifold M . Hence, if M and N are n -dimensional manifolds then we have a few cases to consider. Before we get into them, though, note that $H^0(M\#N) = \mathbb{R} = H^0(M) = H^0(N)$, so $\chi(M) + \chi(N)$ picks up an extra 1 in dimension 0 over $\chi(M\#N)$. Also, we know that for $0 < i < n-1$, $H^i(M\#N) = H^i(M) \oplus H^i(N)$, so the associated Betti number for $M\#N$ is just the sum of the Betti numbers for M and N .

Now, if M and N are both compact and orientable, then we know from problem 5 that, for $0 < i < n$, $H^i(M\#N) = H^i(M) \oplus H^i(N)$. Furthermore, we know that for $i = 0$ or $i = n$, $H^i(M\#N) = H^i(M) = H^i(N)$. Hence, the i th Betti number b_i for $M\#N$ will be the sum of the i th Betti numbers for M and N except when $i = 0$ or $i = n$, when the $M\#N$ Betti number is 1 less than the sum of the Betti numbers for M and N . If n is odd, then these “missing” 1’s will pick up opposite signs and cancel, meaning

$$\chi(M\#N) = \chi(M) + \chi(N).$$

On the other hand, if n is even, then these missing ones will reinforce, meaning

$$\chi(M\#N) = \chi(M) + \chi(N) - 2.$$

On the other hand, suppose M and N are both not compact and orientable (that is, both M and N are non-compact or non-orientable or both). Then we know from above that $H^n(M\#N) = 0 = H^n(M) = H^n(N)$; also, since $0 + 0 = 0$, the n th Betti number for $M\#N$ is the sum of the n th Betti numbers for M and N . On the other hand, we’ve shown that

$$H^{n-1}(M\#N) = \mathbb{R} \oplus H^{n-1}(M) \oplus H^{n-1}(N),$$

so the $(n-1)$ st Betti number for $M\#N$ is one greater than the sum of the Betti numbers for M and N . Therefore, if n is odd, this will add in the alternating sum and cancel the missing 1 from dimension 0, so

$$\chi(M\#N) = \chi(M) + \chi(N).$$

On the other hand, if n is even, then this extra 1 will pick up a minus sign, and so

$$\chi(M\#N) = \chi(M) + \chi(N) - 2.$$

Finally, if M is compact and orientable but N is not, then we know that $H^n(M\#N) = 0 = H^n(N)$, but $H^n(M) = \mathbb{R}$, so the n th Betti number for $M\#N$ is one less than the sum of the Betti numbers for M and N .

On the other hand, we know that $H^{n-1}(M\#N) = H^{n-1}(M) \oplus H^{n-1}(N)$. Therefore, if n is odd, then the -1 from the n th Betti number will pick up a positive sign, cancelling the missing 1 from dimension zero, so

$$\chi(M\#N) = \chi(M) + \chi(N).$$

On the other hand, if n is even, then the missing 1's will reinforce, yielding

$$\chi(M\#N) = \chi(M) + \chi(N) - 2.$$

Therefore, we see that the following gives a complete description of $\chi(M\#N)$:

$$\chi(M\#N) = \begin{cases} \chi(M) + \chi(N) & \text{if } n \text{ odd} \\ \chi(M) + \chi(N) - 2 & \text{if } n \text{ even.} \end{cases}$$



7

Let $M \rightarrow M/G$ be a normal cover of manifolds with G finite. Show that in DeRham cohomology $H^*(M/G)$ is isomorphic to $H^*(M)^G$ where $H^*(M)^G$ are the closed forms in $H^*(M)$ which are invariant under the action of G . Is this true if G is not finite?

Proof. Let $\pi : M \rightarrow M/G$ be the covering map. Then we have the induced map $\pi^* : \Omega(M/G) \rightarrow \Omega(M)$. In fact, if $\omega \in \Omega(M/G)$, then

$$g^*(\pi^*(\omega)) = (\pi \circ g)^*(\omega) = \pi^*(\omega)$$

since $\pi : M \rightarrow M/G$ is a normal cover, so we see that $\pi^* : \Omega(M/G) \rightarrow \Omega(M)^G$, the G -invariant forms on M . We want to show that this map is bijective. To that end, suppose $\pi^*(\omega) = 0$. Then

$$0 = \pi^*\omega(p)(Y_1, \dots, Y_n) = \omega(\pi(p))(d\pi Y_1, \dots, d\pi Y_n)$$

for all $p \in M$. Since π is surjective, this implies that $\omega = 0$, so we see that π^* is injective.

To see that π^* is surjective, suppose $\eta \in \Omega(M)^G$. If $\tilde{p} \in M$ such that $\pi(\tilde{p}) = p$, then we define $\omega \in \Omega(M/G)$ by

$$\omega(p)(X_1, \dots, X_n) := \eta(\tilde{p}) \left(\frac{1}{|G|} \sum_{g \in G} d\pi_{g(\tilde{p})}^{-1} X_1, \dots, \frac{1}{|G|} \sum_{g \in G} d\pi_{g(\tilde{p})}^{-1} X_n \right)$$

where the $d\pi_{g(\tilde{p})}^{-1}$ are the local inverses of $d\pi$ based at $g(\tilde{p})$. To see that this is well-defined, suppose $\tilde{p}' \in M$ such that $\pi(\tilde{p}') = p$. Then, since π is a

normal covering, there exists $g_0 \in G$ such that $\tilde{p}' = g_0(\tilde{p})$. Hence,

$$\begin{aligned}
& \eta(\tilde{p}') \left(\frac{1}{|G|} \sum_{g \in G} d\pi_{g(\tilde{p}')}^{-1} X_1, \dots, \frac{1}{|G|} \sum_{g \in G} d\pi_{g(\tilde{p}')}^{-1} X_n \right) \\
&= \eta(g_0(\tilde{p})) \left(\frac{1}{|G|} \sum_{g \in G} d\pi_{g(g_0(\tilde{p}))}^{-1} X_1, \dots, \frac{1}{|G|} \sum_{g \in G} d\pi_{g(g_0(\tilde{p}))}^{-1} X_n \right) \\
&= \eta(\tilde{p}) \left(\frac{1}{|G|} \sum_{g \in G} d\pi_{g(\tilde{p})}^{-1} X_1, \dots, \frac{1}{|G|} \sum_{g \in G} d\pi_{g(\tilde{p})}^{-1} X_n \right) \\
&= \omega(p)(X_1, \dots, X_n)
\end{aligned}$$

since η is G -invariant. Now,

$$\begin{aligned}
\pi^*(\omega)(\tilde{p})(Y_1, \dots, Y_n) &= \omega(\pi(\tilde{p}))(d\pi Y_1, \dots, d\pi Y_n) \\
&= \eta(\tilde{p}) \left(\frac{1}{|G|} \sum_{g \in G} d\pi_{g(\tilde{p})}^{-1} d\pi Y_1, \dots, \frac{1}{|G|} \sum_{g \in G} d\pi_{g(\tilde{p})}^{-1} d\pi Y_n \right) \\
&= \eta(\tilde{p})(Y_1, \dots, Y_n).
\end{aligned}$$

Hence, we see that $\pi^* : \Omega(M/G) \rightarrow \Omega(M)^G$ is an isomorphism. Hence, since the action of G commutes with the coboundary maps d , the homologies of the two chain complexes are the same:

$$\begin{array}{ccccccc}
\dots & \xrightarrow{d} & \Omega^{k-1}(M/G) & \xrightarrow{d} & \Omega^k(M/G) & \xrightarrow{d} & \Omega^{k+1}(M/G) \xrightarrow{d} \dots \\
& & \downarrow \pi^* & & \downarrow \pi^* & & \downarrow \pi^* \\
\dots & \xrightarrow{d_{k-2}^G} & \Omega^{k-1}(M)^G & \xrightarrow{d_{k-1}^G} & \Omega^k(M)^G & \xrightarrow{d_k^G} & \Omega^{k+1}(M)^G \xrightarrow{d_{k+1}^G} \dots
\end{array}$$

That is, $H^k(M/G) = (\ker d_k^G / \text{im } d_k^G)$. However, $(\ker d_k^G / \text{im } d_k^G)$ consists of precisely the closed forms in $H^k(M)$ that are invariant under the action of G , namely $H^k(M)^G$, so we see that $\pi^* : H^k(M/G) \rightarrow H^k(M)^G$ is an isomorphism. □