GEOMETRY HW 8

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1

Compute the cohomology with $\mathbb{Z}$ and $\mathbb{Z}_2$ coefficients for $K \times \mathbb{RP}^4$ where K is the Klein bottle.

Answer: Throughout, we will make use of the Künneth sequence:

$$\begin{tikzcd}
0 \arrow[r] & \bigoplus_{i=0}^{n} H_i(X,R) \otimes H_{n-i}(Y,R) \arrow[r] & H_n(X \times Y, R) \arrow[r] & \bigoplus_{i=0}^{n-1} \mathrm{Tor}(H_i(X,R),\, H_{n-i-1}(Y,R)) \arrow[d] \\
& & & 0.
\end{tikzcd}$$

Since this sequence splits, we know that

$$H_n(X \times Y, R) \simeq \left(\bigoplus_{i=1}^{n} H_i(X,R) \otimes H_{n-i}(Y,R) \right) \oplus \left(\bigoplus_{j=0}^{n-1} \mathrm{Tor}(H_i(X,R),\, H_{n-i-1}(Y,R)) \right).$$

Now, recall that K has the following homologies:

	$\mathbb{Z}$ coefficients	$\mathbb{Z}/2$ coefficients
H_2	0	$\mathbb{Z}/2$
H_1	$\mathbb{Z} \oplus \mathbb{Z}/2$	$\mathbb{Z}/2 \oplus \mathbb{Z}/2$
H_0	$\mathbb{Z}$	$\mathbb{Z}/2$

Also, $\mathbb{RP}^4$ has the following homologies:

	$\mathbb{Z}$ coefficients	$\mathbb{Z}/2$ coefficients
H_4	0	$\mathbb{Z}/2$
H_3	$\mathbb{Z}/2$	$\mathbb{Z}/2$
H_2	0	$\mathbb{Z}/2$
H_1	$\mathbb{Z}/2$	$\mathbb{Z}/2$
H_0	$\mathbb{Z}$	$\mathbb{Z}/2$

Hence, using the Küunneth formulas and the above homologies,

$$\begin{aligned}
H_0(K \times \mathbb{RP}^4, \mathbb{Z}) &= \mathbb{Z} \otimes \mathbb{Z} = \mathbb{Z} \\
H_1(K \times \mathbb{RP}^4, \mathbb{Z}) &= (\mathbb{Z} \otimes \mathbb{Z}/2) \oplus ((\mathbb{Z} \oplus \mathbb{Z}/2) \otimes \mathbb{Z}) \oplus \mathrm{Tor}(\mathbb{Z}, \mathbb{Z}) \\
&= \mathbb{Z} \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2 \\
H_2(K \times \mathbb{RP}^4, \mathbb{Z}) &= ((\mathbb{Z} \oplus \mathbb{Z}/2) \otimes \mathbb{Z}/2) \oplus \mathrm{Tor}(\mathbb{Z}, \mathbb{Z}/2) \oplus \mathrm{Tor}(\mathbb{Z} \oplus \mathbb{Z}/2, \mathbb{Z}) \\
&= \mathbb{Z}/2 \oplus \mathbb{Z}/2 \\
H_3(K \times \mathbb{RP}^4, \mathbb{Z}) &= (\mathbb{Z} \otimes \mathbb{Z}/2) \oplus \mathrm{Tor}(\mathbb{Z} \oplus \mathbb{Z}/2, \mathbb{Z}/2) = \mathbb{Z}/2 \oplus \mathbb{Z}/2 \\
H_4(K \times \mathbb{RP}^4, \mathbb{Z}) &= (\mathbb{Z} \oplus \mathbb{Z}/2) \otimes \mathbb{Z}/2 = \mathbb{Z}/2 \oplus \mathbb{Z}/2 \\
H_5(K \times \mathbb{RP}^4, \mathbb{Z}) &= \mathrm{Tor}(\mathbb{Z} \oplus \mathbb{Z}/2, \mathbb{Z}/2) = \mathbb{Z}/2,
\end{aligned}$$

where I've omitted all terms of the form $R \otimes 0$, $\mathrm{Tor}(R, 0)$, $\mathrm{Tor}(\mathbb{Z}, R)$, etc.

To compute cohomology, we use universal coefficients:

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{Ext}(H_{n-1}(X), G) \arrow[r] & H^n(X, G) \arrow[r] & \mathrm{Hom}(H_n(X), G) \arrow[r] & 0 \end{tikzcd}$$

which splits, so

$$H^n(X, G) = \mathrm{Ext}(H_{n-1}(X), G) \oplus \mathrm{Hom}(H_n(X), G).$$

Hence,

$$\begin{aligned}
H^0(K \times \mathbb{RP}^4, \mathbb{Z}) &= \mathrm{Hom}(\mathbb{Z}, \mathbb{Z}) = \mathbb{Z} \\
H^1(K \times \mathbb{RP}^4, \mathbb{Z}) &= \mathrm{Ext}(\mathbb{Z}, \mathbb{Z}) \oplus \mathrm{Hom}(\mathbb{Z} \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2, \mathbb{Z}) = \mathbb{Z} \\
H^2(K \times \mathbb{RP}^4, \mathbb{Z}) &= \mathrm{Ext}(\mathbb{Z} \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2, \mathbb{Z}) \oplus \mathrm{Hom}(\mathbb{Z}/2 \oplus \mathbb{Z}/2, \mathbb{Z}) = \mathbb{Z}/2 \oplus \mathbb{Z}/2 \\
H^3(K \times \mathbb{RP}^4, \mathbb{Z}) &= \mathrm{Ext}(\mathbb{Z}/2 \oplus \mathbb{Z}/2, \mathbb{Z}) \oplus \mathrm{Hom}(\mathbb{Z}/2 \oplus \mathbb{Z}/2, \mathbb{Z}) = \mathbb{Z}/2 \oplus \mathbb{Z}/2 \\
H^4(K \times \mathbb{RP}^4, \mathbb{Z}) &= \mathrm{Ext}(\mathbb{Z}/2 \oplus \mathbb{Z}/2, \mathbb{Z}) \oplus \mathrm{Hom}(\mathbb{Z}/2 \oplus \mathbb{Z}/2, \mathbb{Z}) = \mathbb{Z}/2 \oplus \mathbb{Z}/2 \\
H^5(K \times \mathbb{RP}^4, \mathbb{Z}) &= \mathrm{Ext}(\mathbb{Z}/2 \oplus \mathbb{Z}/2, \mathbb{Z}) \oplus \mathrm{Hom}(\mathbb{Z}/2, \mathbb{Z}) = \mathbb{Z}/2 \oplus \mathbb{Z}/2 \\
H^6(K \times \mathbb{RP}^4, \mathbb{Z}) &= \mathrm{Ext}(\mathbb{Z}/2, \mathbb{Z}) \oplus 0 = \mathbb{Z}/2
\end{aligned}$$

Similarly,

$$\begin{aligned}
H^0(K \times \mathbb{RP}^4, \mathbb{Z}/2) &= \mathrm{Hom}(\mathbb{Z}, \mathbb{Z}/2) = \mathbb{Z}/2 \\
H^1(K \times \mathbb{RP}^4, \mathbb{Z}/2) &= \mathrm{Ext}(\mathbb{Z}, \mathbb{Z}/2) \oplus \mathrm{Hom}(\mathbb{Z} \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2, \mathbb{Z}/2) = (\mathbb{Z}/2)^3 \\
H^2(K \times \mathbb{RP}^4, \mathbb{Z}/2) &= \mathrm{Ext}(\mathbb{Z} \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2, \mathbb{Z}/2) \oplus \mathrm{Hom}(\mathbb{Z}/2 \oplus \mathbb{Z}/2, \mathbb{Z}/2) = (\mathbb{Z}/2)^4 \\
H^3(K \times \mathbb{RP}^4, \mathbb{Z}/2) &= \mathrm{Ext}(\mathbb{Z}/2 \oplus \mathbb{Z}/2, \mathbb{Z}/2) \oplus \mathrm{Hom}(\mathbb{Z}/2 \oplus \mathbb{Z}/2, \mathbb{Z}/2) = (\mathbb{Z}/2)^4 \\
H^4(K \times \mathbb{RP}^4, \mathbb{Z}/2) &= \mathrm{Ext}(\mathbb{Z}/2 \oplus \mathbb{Z}/2, \mathbb{Z}/2) \oplus \mathrm{Hom}(\mathbb{Z}/2 \oplus \mathbb{Z}/2, \mathbb{Z}/2) = (\mathbb{Z}/2)^4 \\
H^5(K \times \mathbb{RP}^4, \mathbb{Z}/2) &= \mathrm{Ext}(\mathbb{Z}/2 \oplus \mathbb{Z}/2, \mathbb{Z}/2) \oplus \mathrm{Hom}(\mathbb{Z}/2, \mathbb{Z}/2) = (\mathbb{Z}/2)^3 \\
H^6(K \times \mathbb{RP}^4, \mathbb{Z}/2) &= \mathrm{Ext}(\mathbb{Z}/2, \mathbb{Z}/2) = \mathbb{Z}/2
\end{aligned}$$

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2

Using the Δ complex for the orientable surface of genus g and the non-orientable surface of genus g which we discussed in class with only one vertex, describe the cohomology classes and compute the cup product structure in cohomology.

Answer: We use the following Δ complex:

Then $\partial(p) = 0$ and $\partial(a_i) = \partial(b_i) = \partial(c_i) = 0$. Now,

$$\begin{aligned}
\partial(\Delta_1) &= c_1 + a_1 - b_1 \\
\partial(\Delta_2) &= c_2 + b_1 - c_1 \\
\partial(\Delta_3) &= c_2 + a_2 - c_3 \\
\partial(\Delta_4) &= c_2 + b_2 - c_4 \\
\partial(\Delta_5) &= c_5 + a_2 - c_4 \\
&\vdots
\end{aligned}$$

Now, in computing cohomology,

$$\begin{aligned}
(\delta p^*)(a_i) &= p^*(\partial a_i) = p^*(0) = 0 \\
(\delta p^*)(b_i) &= p^*(\partial b_i) = p^*(0) = 0 \\
(\delta p^*)(c_i) &= p^*(\partial c_i) = p^*(0) = 0,
\end{aligned}$$

so p^* is a generator for $H^0(M_g, \mathbb{Z})$. Since p^* is the only possible generator, we see that $H^0(M_g, \mathbb{Z}) = \mathbb{Z}$.

Now,

$$\begin{aligned}
\delta a_1^* &= \Delta_1^* + \Delta_{2g-1}^* \\
\delta a_i^* &= \Delta_{4i-5}^* + \Delta_{4i-3}^* \text{ for } i > 1
\end{aligned}$$

Also,

$$\begin{aligned}
\delta b_1^* &= \Delta_2^* - \Delta_1^* \\
\delta b_i^* &= \Delta_{4i-4}^* + \Delta_{4i-2}^* \text{ for } i > 1
\end{aligned}$$

Finally,

$$\begin{aligned}
\delta c_1^* &= \Delta_1^* - \Delta_2^* \\
\delta c_2^* &= \Delta_2^* + \Delta_3^* \\
\delta c_3^* &= -\Delta_3^* + \Delta_4^* \\
\delta c_4^* &= -\Delta_4^* - \Delta_5^* \\
\delta c_5^* &= \Delta_5^* - \Delta_6^* \vdots
\end{aligned}$$

Then, note that

$$\begin{gathered}
\delta \left(a_1^* - c_1^* + \sum_{i=2}^{2g-1} c_i^* \right) = 0 \\
\delta \left(a_i^* + c_{4i-5}^* + c_{4i-4}^* \right) = 0 \text{ for } i > 1 \\
\delta \left(b_1^* - c_1^* \right) = 0 \\
\delta \left(b_i^* + c_{4i-4}^* + c_{4i-3}^* \right) = 0 \text{ for } i > 1
\end{gathered}$$

Since there are no coboundaries (since $\delta(p^*) = 0$), all these terms are generators of H^1, which must be free. Since there are no other generators, this means that

$$H^1(M_g, \mathbb{Z}) = \mathbb{Z}^{2g}.$$

Finally, note that $\delta(\Delta_1^*) = 0$ and Δ_1^* is not a coboundary, so Δ_1^* is a generator of H^2. Furthermore, from $\delta(c_1^*)$, we know that, in cohomology, $\Delta_1^* = \Delta_2^* = \cdots$, so we see that this is the only generator. Hence, $H^2(M_g, \mathbb{Z}) = \mathbb{Z}$.

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3

Let X be a CW complex with one cell in dimension i for $0 \leq i \leq 4$. What are the possibilities for the homology and cohomology groups with $\mathbb{Z}$ coefficients? What about the ring structure?

Answer: If the CW complex is as given, then $C_i(X) = \mathbb{Z}$ for $0 \leq i \leq 4$, so we have the chain complex

$$0 \to \mathbb{Z} \overset{\partial_4}{\to} \mathbb{Z} \overset{\partial_3}{\to} \mathbb{Z} \overset{\partial_2}{\to} \mathbb{Z} \overset{\partial_1}{\to} \mathbb{Z} \overset{\partial_0}{\to} 0.$$

Hence, each ∂_i is just multiplication by some integer, say $\partial_i = \cdot a_i$, where a_i is just the degree of the attaching map of the single i-cell in the CW complex.

Now, if $a_i \neq 0$, then $\text{im } \partial_i \neq 0$ and, hence, $\ker \partial_{i-1} \neq 0$. The only integer whose multiplication has non-zero kernel is 0, so $a_i \neq 0$ implies that $a_{i-1} = 0$. In this case, then, $H_{i-1}(X, \mathbb{Z}) = \ker \partial_{i-1} / \text{im } \partial_i = \mathbb{Z}/a_i\mathbb{Z}$. If $a_i = 0$, then there is no restriction on a_{i-1}; if $a_{i-1} = 0$, then $H_{i-1}(X, \mathbb{Z}) = \ker \partial_{i-1} / \text{im } \partial_i = \mathbb{Z}$, whereas if $a_{i-1} \neq 0$, then $\ker \partial_{i-1} = 0$, so $H_{i-1} = 0$.

Also, since $H_0 = \mathbb{Z}$, we know that $a_1 = 0$. Therefore, there are really only five possibilities for the homology groups of X, summarized in the following table:

H_0	H_1	H_2	H_3	H_4
$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$
$\mathbb{Z}$	$\mathbb{Z}/a_2$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$
$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}/a_3$	$\mathbb{Z}$	$\mathbb{Z}$
$\mathbb{Z}$	$\mathbb{Z}/a_2$	$\mathbb{Z}$	$\mathbb{Z}/a_4$	0
$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}/a_4$	0

Then, by universal coefficients, the possible cohomology groups are:

H^0	H^1	H^2	H^3	H^4
$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$
$\mathbb{Z}$	0	$\mathbb{Z} \oplus \mathbb{Z}/a_2$	$\mathbb{Z}$	$\mathbb{Z}$
$\mathbb{Z}$	$\mathbb{Z}$	0	$\mathbb{Z} \oplus \mathbb{Z}/a_3$	$\mathbb{Z}$
$\mathbb{Z}$	0	$\mathbb{Z} \oplus \mathbb{Z}/a_2$	0	$\mathbb{Z}/a_4$
$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	0	$\mathbb{Z}/a_4$

In all of the below, $e_1^2 = e_3^2 = 0$, since our coefficients are anticommutative, and a superscript has torsion; i.e. $e_2^{a_2}$ has a_2-torsion. In the first case, we have generators $1, e_1, e_2, e_3, e_4$ (corresponding to the homology groups), so the possible ring structures are $\mathbb{Z}[e_1, e_2, e_3, e_4]$ modded out by the relations: $e_1 e_3 = \alpha e_4$ and $e_2^2 = \beta e_4$ (where α and/or β may be zero).

In the second case, we have two generators e_2 and $e_2^{a_2}$ in the second dimension and generators e_3 and e_4. Then the ring structure is $\mathbb{Z}[e_2, e_2^{a_2}, e_3, e_4]$ modulo the relations $e_2^2 = \alpha e_4$ (where α may be zero) and $(e_2^{a_2})^2 = e_2 e_2^{a_2} = 0$.

In the third case, we have generators e_1, e_3, $e_3^{a_3}$ and e_4. Then the ring structure is $\mathbb{Z}[e_1, e_3, e_3^{a_3}, e_4]$ modulo the relations: $e_1 e_3 = \alpha e_4$ and $e_1 e_3^{a_3} = 0$, since no multiple of e_4 has any torsion.

In the fourth case, we have generators e_2, $e_2^{a_2}$, $e_4^{a_4}$. Then the ring structure is $\mathbb{Z}[e_2, e_2^{a_2}, e_4^{a_4}]$ modulo the relations $e_2^2 = \alpha e_4^{a_4}$,

$$e_2^c = \begin{cases} 0 & \text{if } (a_4, a_2) = 1 \\ \gamma e_4^{a_4} & \text{else, where } \gamma | (a_4, a_2) \end{cases}$$

and

$$e_2 e_2^c = \begin{cases} 0 & \text{if } (a_4, a_2) = 1 \\ \gamma e_4^{a_4} & \text{else, where } \gamma | (a_4, a_2). \end{cases}$$

In the last case, we have generators e_1, e_2 and $e_4^{a_4}$, so the ring structure is $\mathbb{Z}[e_1, e_2, e_4^{a_4}]/(e_2^2 = \alpha e_4^{a_4})$.

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4

Compute the cohomology groups of $\mathbb{RP}^n$ with coefficients in $\mathbb{Z}$, but without using the universal coefficient theorem. Compute the ring structure for the cohomology ring with $\mathbb{Z}_2$ coefficients for $n = 1, 2, 3$.

Answer: We know that the cell complex for $\mathbb{RP}^n$ is the following:

$$0 \to \mathbb{Z} \xrightarrow{\partial_n} \mathbb{Z} \xrightarrow{\partial_{n-1}} \cdots \xrightarrow{\partial_2} \mathbb{Z} \xrightarrow{\partial_1} \mathbb{Z} \xrightarrow{0} 0,$$

where ∂_k is multiplication by 2 if k is even and the zero map if k is odd. Hence, $C^k(\mathbb{RP}^n, \mathbb{Z}) = \text{Hom}(C_n(\mathbb{RP}^n, \mathbb{Z}), \mathbb{Z}) = \text{Hom}(\mathbb{Z}, \mathbb{Z}, =)\mathbb{Z}$. So we have:

$$0 \xleftarrow{\delta_n} \mathbb{Z} \xleftarrow{\delta_{n-1}} \mathbb{Z} \xleftarrow{\delta_{n-2}} \cdots \xleftarrow{\delta_2} \mathbb{Z} \xleftarrow{\delta_1} \mathbb{Z} \xleftarrow{\delta_0} \mathbb{Z} \leftarrow 0,$$

where $\delta_k = \partial_{k+1}^*$. Specifically, if $\alpha \in C^k(\mathbb{RP}^n, \mathbb{Z})$ and $\sigma \in C_{k+1}(\mathbb{RP}^n, \mathbb{Z})$, then

$$(\delta_k \alpha)(\sigma) = \alpha(\partial_{k+1}\sigma),$$

so δ_k is multiplication by 2 if k is odd and the zero map if k is even. Since $H^k(\mathbb{RP}^n, \mathbb{Z}) = \ker \delta_k / \text{im } \delta_{k-1}$, this implies that

$$H^k(\mathbb{RP}^n, \mathbb{Z}) = \begin{cases} \mathbb{Z} & k = 0 \text{ or } k = n \text{ and } n \text{ odd} \\ \mathbb{Z}/2 & k \text{ even}, k \neq 0k \neq n \\ 0 & k \text{ odd}, k \neq n. \end{cases}$$

Now, turning to the cup product for $i = 1, 2, 3$, consider first $\mathbb{RP}^1 = S^1$. Then, by the above calculation, $H^0(\mathbb{RP}^1, \mathbb{Z}) = H^1(\mathbb{RP}^1, \mathbb{Z}) = \mathbb{Z}$, so if α is a generator of H^1, $\alpha \cup \alpha = 0$, so the ring structure is $\mathbb{Z}/2[a]/(a^2)$.

For $\mathbb{RP}^2$, we have the following Δ-complex:

Then $\partial(p) = \partial(p) = 0$, $\partial(a) = q - p$, $\partial(b) = q - p$ and $\partial(c) = 0$. Also, $\partial(\sigma_1) = c + b - a$ and $\partial(\sigma_2) = c + a - b$. Now, in cohomology, if p^*, q^*, a^*, b^*, c^*, σ_1^* and σ_2^* are the duals of all these, then we see that

$$\begin{aligned} \delta(p^*)(a) &= p^*(\partial(a)) = p^*(q - p) = -1 \\ \delta(p^*)(b) &= p^*(\partial(b)) = p^*(q - p) = -1 \\ \delta(p^*)(c) &= p^*(\partial(c)) = p^*(0) = 0 \\ \delta(q^*)(a) &= q^*(\partial(a)) = q^*(q - p) = 1 \\ \delta(q^*)(b) &= q^*(\partial(b)) = q^*(q - p) = 1 \\ \delta(q^*)(c) &= q^*(\partial(c)) = q^*(0) = 0, \end{aligned}$$

, so $\delta p^* = -a^* - b^*$ and $\delta q^* = a^* + b^*$. Hence, $\delta(p^* + q^*) = 0$, so $p^* + q^*$ is a generator of $H^0(\mathbb{RP}^2, \mathbb{Z}/2) = \mathbb{Z}/2$.

Also,

$$\begin{aligned}
\delta a^*(\sigma_1) &= a^*(\partial\sigma_1) = a^*(c + b - a) = -1\\
\delta a^*(\sigma_2) &= a^*(\partial\sigma_2) = a^*(c + a - b) = 1\\
\delta b^*(\sigma_1) &= b^*(\partial\sigma_1) = b^*(c + b - a) = 1\\
\delta b^*(\sigma_2) &= b^*(\partial\sigma_2) = b^*(c + a - b) = -1\\
\delta c^*(\sigma_1) &= c^*(\partial\sigma_1) = c^*(c + b - a) = 1\\
\delta c^*(\sigma_2) &= c^*(\partial\sigma_2) = c^*(c + a - b) = 1,
\end{aligned}$$

so $\delta a^* = \sigma_2^* - \sigma_1^*$, $\delta b^* = \sigma_1^* - \sigma_2^*$ and $\delta c^* = \sigma_1^* + \sigma_2^*$. In $\mathbb{Z}/2$ coefficients, all the minus signs disappear, so we have $\delta a^* = \delta b^* = \delta c^* = \sigma_1^* + \sigma_2^*$. Since $a^* + b^*$ is a coboundary, we see that the only possible generators are $a^* + c^*$ and $b^* + c^*$. Since $a^* = -b^* = b^*$, these are the same, so the generator of $H^1(\mathbb{RP}^2, \mathbb{Z}/2) = \mathbb{Z}/2$ is $a^* + c^*$. Finally, since $\sigma_1^* = \sigma_2^*$ and $\delta\sigma_2^* = 0$, σ_2^* is the generator of $H^2(\mathbb{RP}^2, \mathbb{Z}/2) = \mathbb{Z}/2$. Now, the only product we need to worry about is from the generator of H^1, $a^* + c^*$. Thus,

$$\begin{aligned}
(a^* + c^*) \cup (a^* + c^*)(\sigma_1) &= (a^* + c^*)(c)(a^* + c^*)(b) = 0\\
(a^* + c^*) \cup (a^* + c^*)(\sigma_2) &= (a^* + c^*)(c)(a^* + c^*)(a) = 1,
\end{aligned}$$

so $(a^* + c^*) \cup (a^* + c^*) = \sigma_2^*$. Hence, the ring structure is $\mathbb{Z}/2[a]/(a^3)$.

For $\mathbb{RP}^3$, we use the Δ-complex pictured on the attached sheet, with boundaries and coboundaries as attached (where all computations are mod 2; i.e., signs are irrelevant). Using these coboundary computations, we see that $\delta(p_1^* + p_2^* + p_3^*) = 0$, so $p_1^* + p_2^* + p_3^*$ is a generator for $H^0(\mathbb{RP}^3, \mathbb{Z}/2) = \mathbb{Z}/2$.

Now, $\delta(a_1^* + a_3^* + a_6^* + a_7^*) = 0$ and, since a_3^* does not appear in the coboundary calculations of any of the p_i^*, this is not a coboundary and so $a_1^* + a_3^* + a_6^* + a_7^*$ is a generator of $H^1(\mathbb{RP}^3, \mathbb{Z}/2) = \mathbb{Z}/2$.

$\delta(\sigma_2^* + \sigma_4^* + \sigma_8^*) = 0$. Now, the coboundaries of each of the 1-co-chains contains an even number of summands; since $\sigma_2^* + \sigma_4^* + \sigma_8^*$ has an odd number of summands, this term is not a coboundary and so must be a generator of $H^2(\mathbb{RP}^3, \mathbb{Z}/2) = \mathbb{Z}/2$. Finally, since $\delta(\tau_i^*) = 0$ for $i = 1, 2, 3, 4$ and $\tau_i^* = \tau_j^*$ for all i, j in cohomology, we see that τ_2^* is a generator of $H^3(\mathbb{RP}^3, \mathbb{Z}/2) = \mathbb{Z}/2$. Let $\nVdash = p_1^* + p_2^* + p_3^*$, $\alpha = a_1^* + a_3^* + a_6^* + a_7^*$, let

$\beta = \sigma_2^* + \sigma_4^* + \sigma_8^*$ and $\gamma = \tau_2^*$. Then,

$$\begin{aligned}
\alpha \cup \alpha(\sigma_1) &= \alpha(a_3)\alpha(a_2) = 0\\
\alpha \cup \alpha(\sigma_2) &= \alpha(a_3)\alpha(a_1) = 1\\
\alpha \cup \alpha(\sigma_3) &= \alpha(a_2)\alpha(a_4) = 0\\
\alpha \cup \alpha(\sigma_4) &= \alpha(a_1)\alpha(a_6) = 1\\
\alpha \cup \alpha(\sigma_5) &= \alpha(a_2)\alpha(a_6) = 0\\
\alpha \cup \alpha(\sigma_6) &= \alpha(a_1)\alpha(a_4) = 0\\
\alpha \cup \alpha(\sigma_7) &= \alpha(a_3)\alpha(a_5) = 0\\
\alpha \cup \alpha(\sigma_8) &= \alpha(a_3)\alpha(a_7) = 1
\end{aligned}$$

so $\alpha \cup \alpha = \sigma_2^* + \sigma_4^* + \sigma_8^* = \beta$.

Also,

$$\begin{aligned}
\alpha \cup \beta(\tau_1) &= \alpha(a_3)\beta(\sigma_3) = 0\\
\alpha \cup \beta(\tau_2) &= \alpha(a_3)\beta(\sigma_4) = 1\\
\alpha \cup \beta(\tau_3) &= \alpha(a_3)\beta(\sigma_6) = 0\\
\alpha \cup \beta(\tau_4) &= \alpha(a_3)\beta(\sigma_5) = 0
\end{aligned}$$

so $\alpha \cup \beta = \tau_2^* = \gamma$. Hence, we've shown that the ring structure on $H^*(\mathbb{RP}^3, \mathbb{Z}/2)$ is simply $\mathbb{Z}/2[a]/(a^4)$.

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