ALGEBRA HW 3

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1

(a): Show that $R[x]$ is a flat R-module.

Proof. Consider the set $A = \{1, x, x^2, \ldots\}$. Then certainly A generates $R[x]$ as an R-module. Suppose there is some finite linear combination of elements of A that equals zero. Since a finite linear combination is just a polynomial, this means that some polynomial

$$a_0 + a_1x + \ldots + a_nx^n = 0.$$

However, this implies that each of the a_i's is zero, so we conclude that A is a linearly independent set, and so A forms a basis for $R[x]$ which is, therefore, free. Since free modules are flat, we conclude that $R[x]$ is a flat R-module. □

(b): Show that $R[x, y]/(xy)$ is *not* a flat $R[x]$-module.

Proof. Consider the map $R[x] \to R[x]$ given by

$$f \mapsto xf.$$

Then this map is certainly a homomorphism and, furthermore, if $f, g \in R[x]$ such that f and g map to the same element, then $xf = xg$ and so it must be the case that $f = g$. In other words, this map is an injection. Now, $R[x, y]/(xy) \otimes_{R[x]} \cdot$ takes this injection to the map $R[x, y]/(xy) \otimes_{R[x]} R[x] \to R[x, y]/(xy) \otimes_{R[x]} R[x]$ given by

$$f \otimes g \mapsto f \otimes xg = xf \otimes g.$$

Now, $R[x, y]/(xy) \otimes_{R[x]} R[x] \simeq R[x, y]/(xy)$ and the induced map $R[x, y]/(xy) \to R[x, y]/(xy)$ is given by

$$f \mapsto xf.$$

However, $y \neq 0$ in $R[x, y]/(xy)$, yet $xy = 0$ in $R[x, y]/(xy)$, so this is not an injection. Hence, $R[x, y]/(xy) \otimes_{R[x]} \cdot$ does not preserve this particular injection, and so $R[x, y]/(xy)$ is not a flat $R[x]$-module. □

(c): Let M, N be flat R-modules. Show that $M \oplus N$ and $M \otimes_R N$ are flat R-modules.

Proof. Suppose S, S' are R-modules and $f : S \hookrightarrow S'$ is an injection. Then the functor defined by $M \oplus N \otimes_R \cdot$ takes this map to

$$1 \otimes f : (M \oplus N) \otimes_R S \to (M \oplus N) \otimes_R S'.$$

In turn, this becomes

$$(1 \otimes f) \oplus (1 \otimes f) : (M \otimes_R S) \oplus (N \otimes_R S) \to (M \otimes_R S) \oplus (N \otimes_R S').$$

However, since M and N are flat, $1 \otimes f : M \otimes_R S \to M \otimes_R S'$ and $1 \otimes f : N \otimes_R S \to N \otimes_R S'$ are both injective, so $(1 \otimes f) \oplus (1 \otimes f)$ is also an injection, and so $M \oplus N$ is flat.

With f, S and S' as above, note that since N is flat, the induced map $N \otimes_R S \to N \otimes_R S'$ is injective, and so, since M is also flat, the map induced by $M \otimes_R \cdot$ on *this* map is also injective. Namely,

$$M \otimes_R (N \otimes_R S) \to M \otimes_R (N \otimes_R S).$$

However, since tensoring is associative, this implies that the map induced by $(M \otimes_R N) \otimes_R \cdot$ on $f : S \to S'$, namely

$$1 \otimes f : (M \otimes_R N) \otimes_R S \to (M \otimes_R N) \otimes_R S'$$

is injective. □

(d): Show that if M is a finitely generated projective R-module, then M is a flat R-module.

Proof. Since M is projective, there exists an R-module N such that $M \oplus N = F$ for some free module F. Since F is free, F is certainly flat. Now, suppose S and S' are R-modules and that $\phi : S \hookrightarrow S'$ is an injection. Then, since F is flat, $1 \otimes \phi : F \otimes_R S \hookrightarrow F \otimes_R S'$ is an injection. Since $F = M \oplus N$ and since $(M \oplus N) \otimes S = (M \otimes S) \oplus (N \otimes S)$ (and similarly for S'), this means that

$$(M \otimes S) \oplus (N \otimes S) \hookrightarrow (M \otimes S') \oplus (N \otimes S') \tag{1}$$

is an injection. However, this map is just $(1 \otimes \phi, 1 \otimes \phi)$ where $1 \otimes \phi$ is the map $M \otimes S \to M \otimes S'$ induced by $M \otimes \cdot$. Since (1) is an injection, $M \otimes S \hookrightarrow M \otimes S'$ must also be an injection. Since our choice of S was arbitrary, we see that this holds for all R-modules, and so M is flat. □

(e): Is the $\mathbb{Z}$-module $\mathbb{Q}$ free? torsion free? flat? projective?

Answer: As we saw in Problem Set 1, problem 1(f), $\mathbb{Q}$ is not a free $\mathbb{Z}$-module. Now, suppose $\mathbb{Q}$ has some torsion element $\frac{p}{q}$. That is, there exists non-zero $a \in \mathbb{Z}$ such that $a\frac{p}{q} = 0$. However,

$$0 = a\frac{p}{q} = \frac{ap}{q}$$

implies that $ap = 0$ and so, since $\mathbb{Z}$ is an integral domain, that p is zero; hence, $\frac{p}{q} = 0$. Therefore, we conclude that $\mathbb{Q}$ is torsion-free.

Now, suppose $f : N' \hookrightarrow N$ is an injective map of $\mathbb{Z}$-modules and let $f \otimes 1 : N' \otimes_{\mathbb{Z}} \mathbb{Q} \to N \otimes_{\mathbb{Z}} \mathbb{Q}$ be the map induced by $\cdot \otimes_{\mathbb{Z}} \mathbb{Q}$. Suppose $\sum_{i=1}^{n} n_i' \otimes \frac{p_i}{q_i} \in N' \otimes_{\mathbb{Z}} \mathbb{Q}$ such that

$$\begin{aligned}
0 &= f \otimes 1 \left(\sum_{i=1}^{n} n_i' \otimes \frac{p_i}{q_i} \right) \\
&= \sum_{i=1}^{n} \left[f(n_i') \otimes \frac{p_i}{q_i} \right] \\
&= \sum_{i=1}^{n} \left[f(n_i') \otimes \frac{p_i q_1 \cdots \widehat{q_i} \cdots q_n}{q_1 \cdots q_n} \right] \\
&= \sum_{i=1}^{n} \left[f(n_i') p_i q_1 \cdots \widehat{q_i} \cdots q_n \otimes \frac{1}{q_1 \cdots q_n} \right] \\
&= \left[\sum_{i=1}^{n} f(n_i') p_i q_1 \cdots \widehat{q_i} \cdots q_n \right] \otimes \frac{1}{q_1 \cdots q_n},
\end{aligned}$$

so

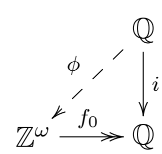
$$0 = \sum_{i=1}^{n} f(n_i') p_1 q_1 \cdots \widehat{q_i} \cdots q_n = f \left(\sum_{i=1}^{n} n_i' p_i q_1 \cdots \widehat{q_i} \cdots q_n \right).$$

Since f is injective, we see that $\sum_{i=1}^{n} n_i' p_i q_1 \cdots \widehat{q_i} \cdots q_n = 0$. Therefore,

$$\begin{aligned}
0 &= \sum_{i=1}^{n} n_i' p_i q_1 \cdots \widehat{q_i} \cdots q_n \otimes \frac{1}{q_1 \cdots q_n} \\
&= \sum_{i=1}^{n} n_i' \otimes \frac{p_i}{q_i}
\end{aligned}$$

so we see that $f \otimes 1$ is injective. Since our choices of N' and N and injection $f : N' \hookrightarrow N$ were arbitrary, this suffices to show that $\mathbb{Q}$ is a flat $\mathbb{Z}$-module.

Now, consider $\mathbb{Q}$ and take the first step in the free resolution of $\mathbb{Q}$, $f_0 : \mathbb{Z}^\omega \twoheadrightarrow \mathbb{Q}$. Let $i : \mathbb{Q} \to \mathbb{Q}$ be the identity map. Then we have the following diagram:



If we can show there is no map $\phi : \mathbb{Q} \to \mathbb{Z}^\omega$ (as indicated by the dashed arrow in the above diagram) such that the diagram commutes, then this suffices to show that $\mathbb{Q}$ is not projective. Suppose

$\phi : \mathbb{Q} \to \mathbb{Z}^\omega$ were such a map. Then $\phi(1) = (a_1, \ldots, a_n, 0, \ldots)$ for some $a_1, \ldots, a_n \in \mathbb{Z}$. Let $a = \max\{a_i\}$. Then

$$(a_1, \ldots, a_n, 0, \ldots) = \phi(1) = \phi(\frac{a+1}{a+1}) = (a+1)\phi(\frac{1}{a+1}).$$

Hence, $a+1$ must divide each of $a_1, \ldots, a_n$. However, since $a+1 > a_i$ for all $i = 1, \ldots, n$, this is impossible, so we see that there is no such ϕ and, therefore, $\mathbb{Q}$ is not a projective $\mathbb{Z}$-module.

♣

2

Suppose that $0 \to M' \xrightarrow{f} M \xrightarrow{g} M'' \to 0$ is an exact sequence of R-modules. Let $M_1 \subset M_2 \subset \cdots$ be a chain of submodules of M, and define $M_i' = f^{-1}(M_i)$ and $M_i'' = g(M_i)$.

(a): Show that $M_1' \subset M_2' \subset \cdots$ is a chain of submodules of M'.

Proof. Note that, if $m' \in M_i'$, then $f(m') \in M_i \subset M_{i+1}$, so $m' \in M_{i+1}'$. Since our choice of i was arbitrary, we see that

$$M_1' \subset M_2' \subset \cdots$$

Now, if $m_1, m_2 \in M_i'$ and $c_1, c_2 \in R$, then

$$f(c_1m_1 + c_2m_2) = c_1f(m_1) + c_2f(m_2) \in M_i,$$

so $c_1m_1 + c_2m_2 \in M_i'$. Furthermore, if $m \in M_i'$, then $f(-m) = -f(m) \in M_i$, so $-m \in M_i'$. Since M_i' inherits all other necessary properties to be a module from M_{i+1}', we see that M_i' is a submodule of M' for all i, so $M_1' \subset M_2' \subset \cdots$ is a chain of submodules of M' □

(b): Show that $M_1'' \subset M_2'' \subset \cdots$ is a chain of submodules of M''.

Proof. If $m'' \in M_i'$, then $m'' = g(m)$ for some $m \in M_i \subset M_{i+1}$. Hence, $g(m) = m'' \in M_{i+1}''$, so $M_1'' \subset M_2'' \subset \cdots$. Now, if $m_1'', m_2'' \in M_i''$ and $c_1, c_2 \in R$, then there exist $m_1, m_2 \in M_i$ such that $g(m_1) = m_1''$ and $g(m_2) = m_2''$. Now,

$$g(c_1m_1 + c_2m_2) = c_1g(m_1) + c_2g(m_2) = c_1m_1'' + c_2m_2'',$$

so $c_1m_1'' + c_2m_2'' \in M_i''$. Furthermore, if $m'' = g(m) \in M_i''$, then $-m \in M_i$ since M_i is a submodule, so

$$g(-m) = -g(m) = -m'',$$

so $-m'' \in M_i''$. Since M_i'' inherits all other necessary properties of a module from M'', we see that $M_1'' \subset M_2'' \subset \cdots$ is a chain of submodules of M''. □

(c): Show that if $i < j$, then the inclusion map $M_i \hookrightarrow M_j$ induces inclusions $M'_i \hookrightarrow M'_j$ and $M''_i \hookrightarrow M''_j$ and also the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M'_i & \longrightarrow & M_i & \longrightarrow & M''_i & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & M'_j & \longrightarrow & M_j & \longrightarrow & M''_j & \longrightarrow & 0 \end{array}$$

Proof. If $i_{ij} : M_i \hookrightarrow M_j$ is the inclusion map, then, for any $m' \in M'_i \subset M'_j$, we also have the inclusion $i'_{ij} : M'_i \hookrightarrow M'_j$ given by $i'_{ij}(m') = m'$. Furthermore,

$$f \circ i'_{ij}(m') = f(m') = i_{ij} \circ f(m').$$

On the other hand, for any $m'' \in M''_i \subset M''_j$ we have the inclusion $i''_{ij} : M''_i \hookrightarrow M''_j$ given by $i''_{ij}(m'') = m''$. Now, if $m \in M_i$, then

$$g \circ i_{ij}(m) = g(m) = i''_{ij} \circ g(m).$$

Now consider the rows $0 \to M'_i \xrightarrow{f} M_i \xrightarrow{g} M''_i \to 0$. f is injective on all of M' and therefore on $M'_i \subset M'$. g is surjective on all of M and therefore on $M_i \subset M$. If $m \in \ker g|_{M_i}$, then $m \in \ker g = \text{im } f$, so $m = f(m')$ for some $m' \in M'$. However, since $M'_i = f^{-1}(M_i)$, $m' \in M'_i$, so $\ker g|_{M_i} \subset \text{im } f|_{M_i}$. On the other hand, if $m' \in M'_i$, then $g|_{M_i} \circ f|_{M_i}(m') = g \circ f(m') = 0$, so $\text{im } f|_{M_i} \subset \ker g|_{M_i}$ and, therefore, we see that this sequence is exact.

What we've just shown is exactly that

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M'_i & \longrightarrow & M_i & \longrightarrow & M''_i & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & M'_j & \longrightarrow & M_j & \longrightarrow & M''_j & \longrightarrow & 0 \end{array}$$

is a commutative diagram with exact rows. □

3

In the notation of Problem Set 1, problems 4 and 5:

(a): Find linear polynomials $f, g \in R$ such that the only maximal ideal of R containing f is I, and the only maximal ideal of R containing g is J.

Answer: Suppose $f \in R$ is a linear polynomial such that the only maximal ideal of R containing f is I. Then f can be represented by a linear polynomial $F \in \mathbb{R}[x, y]$ and, since the only maximal ideal of R containing f is I, the only point at which $F = 0$ can intersect the circle $x^2 + y^2 = 25$ is at the point $(3, 4)$; that is, F is the tangent line to the circle at this point. Implicitly differeniating the equation of the circle, $2x + 2y\frac{dy}{dx} = 0$, so the slope of F at $(3,4)$ is

$$\frac{dy}{dx} = \frac{-2x}{2y} = \frac{-6}{8} = \frac{-3}{4}.$$

Hence, $F(x,y) = 3x + 4y + C$ for some C such that $F(3,4) = 0$; therefore, $F(x,y) = 3x + 4y - 25$. Restricting to R, we conclude that $f(x,y) = 3x+4y-25 \in R$ is a linear polynomial such that the only maximal ideal of R containing it is I.

On the other hand, suppose $g \in R$ is a linear polynomial such that the only maximal ideal of R containing g is J. Then g can be represented by a linear polynomial $G \in \mathbb{R}[x,y]$ such that $G = 0$ only intersects the circle at the point $(-3,4)$, so $G = 0$ is a tangent line to the circle at this point. Since this point is just a reflection across the y-axis of the point $(3,4)$, we see that the slope of the tangent line is $-\frac{-3}{4} = \frac{3}{4}$, so $G(x,y) = -3x+4y+C$ for C such that $G(-3,4) = 0$; hence, $G(x,y) = -3x+4y-25$, and so, restricting to R, $g(x,y) = -3x+4y-25$ is a linear polynomial such that the only maximal ideal of R containing g is J.

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(b): Find a linear polynomial $h \in R$ such that $h \in J$ and $h \in K$, where K is the maximal ideal corresponding to the point $S = (-3,4)$.

Answer: If $h \in R$ such that $h \in J$ and $h \in K$, then h must be represented by some $H \in \mathbb{R}[x,y]$ such that $H = 0$ intersects the circle in the points $(-3,4)$ and $(3,-4)$. Now, the slope of H must be

$$\frac{-4-4}{3+3} = \frac{-8}{6} = \frac{-4}{3};$$

so $H(x,y) = 4x+3y+C$ for some C such that $H(-3,4) = 0$; hence, $C = 0$, and so $H(x,y) = 4x + 3y$ (note that $4(3) + 3(-4) = 0$ as well, so this line really does pass through the two desired points). Therefore, restricting to R, $h(x,y) = 4x+3y$ is a linear polynomial contained in J and K.

♣

(c): Show that $I_f \stackrel{\text{def}}{=} I \otimes_R R[\frac{1}{f}]$ is a free $R[\frac{1}{f}]$-module, viz. is the unit ideal in $R[\frac{1}{f}]$.

Proof. Define the map $\phi : I \times_R R[\frac{1}{f}] \to R[\frac{1}{f}]$ by

$$(i,b) \mapsto ib.$$

Then for $i_1, i_2 \in I$, $b_1, b_2 \in R[\frac{1}{f}]$ and $c_1, c_2 \in R$,

$$\begin{aligned}\phi((c_1 i_1, b_1) + (c_2 i_2, b_1)) = \phi((c_1 i_1 + c_2 i_2, b_1)) &= (c_1 i_1 + c_2 i_2) b_1 \\ &= c_1 i_1 b_1 + c_2 i_2 b_1 \\ &= c_1 \phi((i_1, b_1)) + c_2 \phi((i_2, b_1))\end{aligned}$$

and

$$\begin{aligned}\phi((i_1, c_1 b_1) + (i_2, c_2 b_1)) = \phi((i_1, c_1 b_1 + c_2 b_2)) &= i_1(c_1 b_1 + c_2 b_2) \\ &= c_1 i_1 b_1 + c_2 i_1 b_2 \\ &= c_1 \phi((i_1, b_1)) + c_2 \phi((i_1, b_2)),\end{aligned}$$

so ϕ is bilinear and, therefore, induces a unique homomorphism $\Phi : I \otimes_R R[\frac{1}{f}] \to R[\frac{1}{f}]$ such that $\Phi(i \otimes b) = ib$. Now, define $\Psi : R[\frac{1}{f}] \to I \otimes_R R[\frac{1}{f}]$ by

$$a \mapsto af \otimes \frac{1}{f}.$$

Then, for $a_1, a_2 \in R[\frac{1}{f}]$ and $c_1, c_2 \in R$,

$$\Psi(c_1 a_1 + c_2 a_2) = (c_1 a_1 + c_2 a_2) f \otimes \frac{1}{f} = c_1 a_1 f \otimes \frac{1}{f} + c_2 a_2 f \otimes \frac{1}{f} = c_1 \Psi(a_1) + c_2 \Psi(a_2),$$

so Ψ is a homomorphism. Now,

$$\Psi \circ \Phi(i \otimes b) = \Psi(ib) = ibf \otimes \frac{1}{f} = if \otimes \frac{b}{f} = i \otimes b$$

and

$$\Phi \circ \Psi(a) = \Phi(af \otimes \frac{1}{f}) = af \frac{1}{f} = a.$$

Hence, Φ and Ψ are inverses of eachother, and so $\Phi : I_f = I \otimes_R R[\frac{1}{f}] \to R[\frac{1}{f}]$ is an isomorphism. Thus, I_f is a free $R[\frac{1}{f}]$-module. □

(d): Show that for suitable choices of g, h above, $\frac{x-3}{y-4} = \frac{h}{g}$ in R. Explain this equality geometrically, in terms of the graphs of $g = 0$, $h = 0$, $x - 3 = 0$, $y - 4 = 0$, and $x^2 + y^2 = 25$.

Proof. Given the choices of g and h above, consider:

$$(x-3)g = (x-3)(-3x + 4y - 25) = -3x^2 - 16x + 4xy - 12y + 75.$$

On the other hand,

$$\begin{aligned}(y-4)h = (y-4)(4x+3y) &= -16x + 4xy - 12y + 3y^2 \\ &= -16x + 4xy - 12y + 3(25 - x^2) \\ &= -16x + 4xy - 12y + 75 - 3x^2 \\ &= (x-3)g,\end{aligned}$$

since $x^2 + y^2 - 25 = 0$ in R. Hence, since $(x-3)g = (y-4)h$. Now, since $g \in J$, we saw on Problem Set 1 #5(a) that

$$\frac{x-3}{y-4} g \in R,$$

and so it is legitimate to say that $\frac{x-3}{y-4} g = h$ and so $\frac{x-3}{y-4} = \frac{h}{g}$ in R.

Now, consider the following picture of $x-3=0$, $y-4=0$, $g=0$, $h=0$ and $x^2+y^2-25=0$:

If we invert $y-4$, this kills off the zero locus of $y-4$; specifically, $\frac{x-3}{y-4}$ will correspond, in R, to the point only point away from this locus at which $x-3=0$ intersects the circle, namely $(3,-4)$. On the other hand, inverting g kills of the zero locus of g; in R, then, $\frac{h}{g}$ will correspond to the only point away from this locus at which $h=0$ intersects the circle. By construction, this is exactly the point $(3,-4)$, so it should come as no surprise that $\frac{x-3}{y-4} = \frac{h}{g}$ in R. □

(e): Using (d), show that $I_g \stackrel{\text{def}}{=} I \otimes_R R[\frac{1}{g}]$ is a free $R[\frac{1}{g}]$-module, viz is the ideal $(y-4)$ in $R[\frac{1}{g}]$.

Proof. Define $\phi : I \times R[\frac{1}{g}] \to (y-4)$ (where $(y-4)$ is considered as an ideal in $R[\frac{1}{g}]$) by

$$(i, b) \mapsto ib.$$

Now, we need to know that $ib \in (y-4)$ to be sure that this really is a valid map. However, we know that $i = (x-3)p + (y-4)q$ for some $p, q \in R$, and so, using (d) above,

$$\begin{aligned} ib = ((x-3)p + (y-4)q)b &= ((x-3)p + (y-4)q)\frac{bg}{g} \\ &= \frac{b}{g}((x-3)gp + (y-4)gq) \\ &= \frac{b}{g}((y-4)hp + (y-4)gq) \\ &= \frac{b}{g}(hp+gq)(y-4) \in (y-4), \end{aligned}$$

so this is indeed a valid map. Now, if $i_1, i_2 \in I$, $b_1, b_2 \in R[\frac{1}{g}]$ and $c_1, c_2 \in R$, then

$$\begin{aligned} \phi((c_1 i_1, b_1) + (c_2 i_2, b_1)) = \phi((c_1 i_1 + c_2 i_2, b_1)) &= (c_1 i_1 + c_2 i_2) b_1 \\ &= c_1 i_1 b_1 + c_2 i_2 b_1 \\ &= c_1 \phi((i_1, b_1)) + c_2 \phi((i_2, b_1)) \end{aligned}$$

and

$$\begin{aligned} \phi((i_1, c_1 b_1) + (i_2, c_2 b_1)) = \phi((i_1, c_1 b_1 + c_2 b_2)) &= i_1(c_1 b_1 + c_2 b_2) \\ &= c_1 i_1 b_1 + c_2 i_1 b_2 \\ &= c_1 \phi((i_1, b_1)) + c_2 \phi((i_1, b_2)), \end{aligned}$$

so ϕ is bilinear and, therefore, induces a unique homomorphism $\Phi : I \otimes_R R[\frac{1}{g}] \to (y-4)$ such that $\Phi(i \otimes b) = ib$. Now, define $\Psi : R[\frac{1}{g}] \to I \otimes_R R[\frac{1}{g}]$ by

$$a(y-4) \mapsto a(y-4) \otimes 1.$$

Then, for $a_1, a_2 \in R[\frac{1}{g}]$ and $c_1, c_2 \in R$,

$$\begin{aligned} \Psi(c_1 a_1 (y-4) + c_2 a_2 (y-4)) = \Psi((c_1 a_1 + c_2 a_2)(y-4)) &= (c_1 a_1 + c_2 a_2)(y-4) \otimes 1 \\ &= c_1 a_1 (y-4) \otimes 1 + c_2 a_2 (y-4) \otimes 1 \\ &= c_1 \Psi(a_1(y-4)) + c_2 \Psi(a_2(y-4)), \end{aligned}$$

so Ψ is a homomorphism. Now,

$$\Psi \circ \Phi(i \otimes b) = \Psi(ib) = ib \otimes 1 = i \otimes b$$

and

$$\Phi \circ \Psi(a) = \Phi(a \otimes 1) = a,$$

so Φ and Ψ are inverses, and so we see that $\Phi : I_g \to (y-4)$ is an isomorphism; since $(y-4)$ is a free $R[\frac{1}{g}]$-module, this implies that I_g is free. $\square$

4

(a): Let R be a commutative ring and suppose that *every* R-module M is free. Show that R is a field.

Proof. Let $r \in R$ be non-zero and consider $R/(r)$. Then $R/(r)$ is an R-module and therefore, by hypothesis, free. However, for any $\overline{a} \in R/(r)$ with representative a, $ar \in (r)$, and so $\overline{a}r = 0$ in $R/(r)$. Since $R/(r)$ is free, $R/(r)$ cannot have r-torsion, so it must have been the case that $\overline{a} = 0$ in $R/(r)$. However, our choice of $\overline{a}$ was arbitrary, so we see that there are no non-zero elements of $R/(r)$. Hence, $(r) = R = (1)$, meaning that r is a unit in R. However, since our choice of non-zero r was arbitrary, this implies that every non-zero element of R is a unit, and so R is a field. □

(b): Let $R = \mathbb{R}[x, y]/(x^2 + y^2 - 25)$. Is R a PID? Is every finitely generated projective R-module free?

Answer: In Problem Set 1 #4(c) we concluded that the ideal I of R is not principal, so R is not a PID. In PS1#4(e) we concluded that I is not a free module, whereas in PS1#5(c) we determined that $I \oplus J$ is free, meaning that I is the direct summand of a free module and, therefore, projective. However, in PS1#4(a) we saw that I is generated by $x - 3$ and $y - 4$, so I is finitely generated and so I is an example of a finitely generated projective R-module that is not free.

♣

5

Let R be a commutative ring, and let M, N, S be R-modules. Assume that M is finitely presented and that S is flat. Consider the natural map

$$\alpha : S \otimes_R \mathrm{Hom}(M,\, N) \to \mathrm{Hom}(M,\, S \otimes_R N)$$

taking $s \otimes \phi$ (for $s \in S$ and $\phi \in \hom MN$) to the homomorphism $m \mapsto s \otimes \phi(m)$.

(a): Show that if M is a free R-module then α is a homomorphism.

Proof. Since M is a finitely presented free R-module, $M = R^n$ for some $n \in \mathbb{N}$. Now, if $\phi \in \mathrm{Hom}(M,\, N)$, then $\phi : R^n \to N$ is completely determined by $\phi(e_i)$ for basis elements $e_1, \ldots, e_n$. Since there are no restrictions on the $\phi(e_i)$, we see that $\mathrm{Hom}(M,\, N) \simeq N^n$ (note that this result was independent of any properties of N). Note also that we can denote ϕ simply by a tuple $(n_1, \ldots, n_n)$, where the $n_i \in N$ and $\phi(e_i) = n_i$. Hence,

$$S \otimes_R \mathrm{Hom}(M,\, N) \simeq S \otimes_R N^n \simeq (S \otimes_R N)^n$$

since the tensor product commutes with the direct sum. This isomorphism is given by

$$s \otimes \phi \mapsto (s \otimes n_1, \ldots, s \otimes n_n),$$

where ϕ is represented by $(n_1, \ldots, n_n)$ as above.

On the other hand, by the same argument as above, $\mathrm{Hom}(M,\, S \otimes_R N) = \mathrm{Hom}(R^n,\, S \otimes_R N) \simeq (S \otimes_R)^n$. As above, if $\phi \in \mathrm{Hom}(M,\, S \otimes_R N)$, then ϕ can be represented by

$$(s_1 \otimes n_1, \ldots, s_n \otimes n_n)$$

where $s_i \otimes n_i \in S \otimes_R N$ and $\phi(e_i) = s_i \otimes n_i$. Since both sides are isomorphic to $(S \otimes_R)^n$, it is at least plausible that α is an isomorphism.

Now, under the isomorphisms described above, if $(s \otimes n_1, \ldots, s \otimes n_n) \in (S \otimes N)^n$ on the lefthand side, then this corresponds to the element $s \otimes (e_i \mapsto n_i) \in S \otimes_R \mathrm{Hom}(M,\, N)$; under α,

$$\alpha : s \otimes (e_i \mapsto n_i) \mapsto (e_i \mapsto s \otimes n_i),$$

which, under the correspondence described above, corresponds to the element $(s \otimes n_1, \ldots, s \otimes n_n) \in (S \otimes_R N)^n$ on the right. Hence, we see that α induces the identity map $id : (S \otimes_R N)^n \to (S \otimes_R N)^n$, which is, obviously, an isomorphism. Therefore, composing in reverse of how we did in the above argument, we see that, in fact, α is an isomorphism. □

(b): Suppose more generally that $R^a \xrightarrow{f_a} R^b \xrightarrow{f_b} M \to 0$ is a finite presentation for M. Show that the induced diagram

(2)

$$\begin{array}{ccccccc} 0 \longrightarrow & S \otimes_R \mathrm{Hom}(M,\, N) & \longrightarrow & S \otimes_R \mathrm{Hom}(R^b,\, N) & \longrightarrow & S \otimes_R \mathrm{Hom}(R^a,\, N) \\ & \downarrow & & \downarrow & & \downarrow \\ 0 \longrightarrow & \mathrm{Hom}(M,\, S \otimes_R N) & \longrightarrow & \mathrm{Hom}(R^b,\, S \otimes_R N) & \longrightarrow & \mathrm{Hom}(R^a,\, S \otimes_R N) \end{array}$$

is commutative and has exact rows.

Proof. Since $\mathrm{Hom}(\cdot,\, N)$ is a contravariant left exact functor, applying it to the free resolution $R^a \xrightarrow{f_a} R^b \xrightarrow{f_b} M \to 0$ yields the exact sequence

$$0 \to \mathrm{Hom}(M,\, N) \xrightarrow{(f_b)_*} \mathrm{Hom}(R^b,\, N) \xrightarrow{(f_a)_*} \mathrm{Hom}(R^a,\, N).$$

Since S is flat, $S \otimes_R \cdot$ is exact, so applying this functor to this sequence yields the exact sequence

$$0 \to S \otimes_R \mathrm{Hom}(M,\, N) \xrightarrow{1 \otimes (f_b)_*} S \otimes_R \mathrm{Hom}(R^b,\, N) \xrightarrow{1 \otimes (f_a)_*} S \otimes_R \mathrm{Hom}(R^a,\, N).$$

Similarly, since $\mathrm{Hom}(\cdot, S \otimes_R N)$ is a contravariant left exact functor, applying it to the free resolution yields the exact sequence

$$0 \to \mathrm{Hom}(M, S \otimes_R N) \stackrel{(f_b)'_*}{\to} \mathrm{Hom}(R^b, S \otimes_R N) \stackrel{(f_a)'_*}{\to} \mathrm{Hom}(R^a, S \otimes_R N).$$

Thus, the rows in (2) are exact.

Now, consider the square

$$\begin{array}{ccc} S \otimes_R \mathrm{Hom}(M, N) & \xrightarrow{1\otimes(f_b)_*} & S \otimes_R \mathrm{Hom}(R^b, N) \\ \downarrow \alpha & & \downarrow \alpha' \\ \mathrm{Hom}(M, S \otimes_R N) & \xrightarrow{(f_b)'_*} & \mathrm{Hom}(R^b, S \otimes_R N) \end{array}$$

where $\alpha' : S \otimes_R \mathrm{Hom}(R^b, N) \to \mathrm{Hom}(R^b, S \otimes_R N)$ is defined in a parallel fashion to α. Then, if $s \otimes \phi \in S \otimes_R \mathrm{Hom}(M, N)$,

$$\alpha' \circ (1 \otimes (f_b)_*)(s \otimes \phi) = \alpha'(s \otimes f_b \circ \phi) = (m \mapsto s \otimes f_b \circ \phi(m)).$$

On the other hand,

$$(f_b)'_* \circ \alpha(s \otimes \phi) = (f_b)'_*(m \mapsto s \otimes \phi(m)) = (m \mapsto s \otimes f_b \circ \phi(m)),$$

so we see that this square commutes. A similar argument demonstrates that the other square (2) commutes, so we conclude that (2) is a commutative diagram with exact rows. □

(c): Using the Five Lemma and part (a), deduce that α is an isomorphism.

Proof. If α'' is the map $S \otimes_R \mathrm{Hom}(R^a, N) \to \mathrm{Hom}(R^a, S \otimes_R N)$ defined in a parallel fashion to α and α'' then, by part (a) above, α' and α'' are isomorphisms. Hence, by adding zeros on the left, we have the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \to & 0 & \to & S \otimes_R \mathrm{Hom}(M, N) & \xrightarrow{1\otimes(f_b)_*} & S \otimes_R \mathrm{Hom}(R^b, N) & \xrightarrow{1\otimes(f_a)_*} & S \otimes_R \mathrm{Hom}(R^a, N) \\ \| & & \| & & \downarrow \alpha & & \downarrow \alpha' & & \downarrow \alpha'' \\ 0 & \to & 0 & \to & \mathrm{Hom}(M, S \otimes_R N) & \xrightarrow{(f_b)'_*} & \mathrm{Hom}(R^b, S \otimes_R N) & \xrightarrow{(f_a)'_*} & \mathrm{Hom}(R^a, S \otimes_R N) \end{array}$$

Since α' and α'' are isomorphisms and the identity maps on the left are isomorphisms, we conclude, by the 5 lemma, that α is an isomorphism. □

6

Let $\phi : \mathbb{Z}^2 \to \mathbb{Z}^3$ be given by the matrix

$$\begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix}$$

and let M be the cokernel of ϕ.

(a): Find all $n \in \mathbb{Z}$ such that the $\mathbb{Z}$-module M has (non-zero) n-torsion.

Answer: Since M is the cokernel of ϕ, $M \simeq \mathbb{Z}^3/(\text{im } \phi)$. Now, since the given matrix is similar to the matrix

$$\begin{pmatrix} 1 & 0 \\ 0 & 3 \\ 0 & 0 \end{pmatrix},$$

we see that, after a suitable change of basis of $\mathbb{Z}^2$, the image of ϕ in $\mathbb{Z}^3$ is simply the span of the set $\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 3 \\ 0 \end{pmatrix} \right\}$. Hence, if e_1, e_2, e_3 are the standard basis vectors for $\mathbb{Z}^3$, we see that $M \simeq \mathbb{Z}^3/\langle e_1, 3e_2 \rangle$, so M has the presentation

$$\langle e_1, e_2, e_3 \rangle / \langle e_1, 3e_2 \rangle = \langle e_2, e_3 \rangle / \langle 3e_2 \rangle \simeq \mathbb{Z} \oplus \mathbb{Z}/3.$$

Thus, we see that the only torsion in M is $3k$-torsion for $k \in \mathbb{N}$, since $e_2 \in M$ but $3ke_2 = 0$ in M.

♣

(b): Is M free? flat? torsion free? projective?

Answer: As we've just seen, M is not torsion free and, therefore, cannot be free. Now, since flat, free and projective are all the same thing in finitely generated modules over a PID and $\mathbb{Z}$ is a PID, M is not flat or projective, either.

♣

(c): Show that M has a finite free resolution.

Proof. To construct the free resolution of M, first we take the free module on the generators of M; since the generators are just e_2 and e_3, this is just $\langle e_2, e_3 \rangle \simeq \mathbb{Z}^2$. Let e_1', e_2' be the standard basis elements of $\mathbb{Z}^2$. Then under the map $\mathbb{Z}^2 \to M$, $e_1' \mapsto e_2$ and $e_2' \mapsto e_3$. Since the only relation in M is $3e_2 = 0$, the kernel of this map is simply $K = \langle 3e_1' \rangle \simeq 3\mathbb{Z}$. This is already a free module, so there will be no kernel when we take the free module on the generators of K, namely $\mathbb{Z}$ and define the projection map $1 \mapsto 3e_1'$. Hence, the free resolution of M is simply:

$$0 \to \mathbb{Z} \xrightarrow{3\cdot} \mathbb{Z}^2 \to M \to 0,$$

which is certainly a finite free resolution. □

(d): For each prime number p, compute $M \otimes \mathbb{Z}/p = \text{Tor}^0(M, \mathbb{Z}/p)$ and $\text{Tor}^1(M, \mathbb{Z}/p)$.

Answer: Since $M \simeq \mathbb{Z} \oplus \mathbb{Z}/3$,

$$M \otimes \mathbb{Z}/p \simeq (\mathbb{Z} \oplus \mathbb{Z}/3) \otimes \mathbb{Z}/p = (\mathbb{Z} \otimes \mathbb{Z}/p) \oplus (\mathbb{Z}/3 \otimes \mathbb{Z}/p) = \mathbb{Z}/p \oplus (\mathbb{Z}/3 \otimes \mathbb{Z}/p).$$

Now, if $p \neq 3$, $\mathbb{Z}/3 \otimes \mathbb{Z}/p = 0$, so the above tells us that $\text{Tor}^0(M, \mathbb{Z}/p) = M \otimes \mathbb{Z}/p \simeq \mathbb{Z}/p$. On the other hand, if $p = 3$, then $\mathbb{Z}/3 \otimes \mathbb{Z}/p = \mathbb{Z}/3 \otimes \mathbb{Z}/3 = \mathbb{Z}/3$, so $\text{Tor}^0(M, \mathbb{Z}/p) = M \otimes \mathbb{Z}/p = \mathbb{Z}/3 \oplus \mathbb{Z}/3$.

To compute $\text{Tor}^1(M, \mathbb{Z}/p)$ consider the free resolution computed in (c) above:

$$0 \to \mathbb{Z} \xrightarrow{3\cdot} \mathbb{Z}^2 \to M \to 0.$$

Then, applying $\cdot \otimes \mathbb{Z}/p$ yields:

$$0 \longrightarrow \mathbb{Z} \otimes \mathbb{Z}/p \xrightarrow{d_1} \mathbb{Z}^2 \otimes \mathbb{Z}/p \longrightarrow 0$$

Now $\text{Tor}^1(M, \mathbb{Z}/p) = \ker d_1$. Note that $\mathbb{Z} \otimes \mathbb{Z}/p = \mathbb{Z}/p$ and $\mathbb{Z}^2 \otimes \mathbb{Z}/p = \mathbb{Z}/p \oplus \mathbb{Z}/p$ and recall that $d_1 = 3 \cdot \otimes 1$, so we can reduce the above to

$$0 \longrightarrow \mathbb{Z}/p \xrightarrow{(3\cdot,0)} \mathbb{Z}/p \oplus \mathbb{Z}/p \longrightarrow 0$$

Hence, if $a \in \mathbb{Z}/p$ is in the kernel of d_1, then $(3a, 0) = 0$ in $\mathbb{Z}/p \oplus \mathbb{Z}/p$. Hence, $3a|p$. If $p \neq 3$, then this means that $a = 0$, so $\text{Tor}^1(M, \mathbb{Z}/p) \ker d_1 = 0$ for $p \neq 3$. If $p = 3$, then $\text{Tor}^1(M, \mathbb{Z}/p) = \ker d_1 \simeq \mathbb{Z}/3$.

♣

(e): For every $\mathbb{Z}$-module N and every $i \geq 2$, compute $\text{Tor}^i(M, N)$.

Answer: Since the free resolution of M ends after the second stage (that is, if we denote the free resolution by $\cdots \to F_2 \to F_1 \to F_0 \to M \to 0$, then $F_i = 0$ for $i \geq 2$), we see that $\text{Tor}^i(M, N) = \ker d_i/\text{im } d_{i+1} = 0$ since $\ker d_i = 0 = \text{im } d_{i+1}$ (since $F_i = 0$) for all $i \geq 2$.

♣

7

Let M, N be R-modules, and let $0 \to N \to I_0 \xrightarrow{f_0} I_1 \xrightarrow{f_1} I_2 \xrightarrow{f_2} \cdots$ be an injective resolution of N. Let $\phi \in \text{Ext}^1(M, N)$, and choose a homomorphism $\Phi \in \text{Hom}(M, I_1)$ representing ϕ (where we used the above resolution to compute Ext).

(a): Show that $f_1 \circ \Phi = 0$, and deduce that $\Phi : M \to I_0/N$.

Proof. Under $\text{Hom}(M, \cdot)$ and removing the N term, the injective resolution becomes

$$0 \longrightarrow \text{Hom}(M, I_0) \xrightarrow{\partial_0} \text{Hom}(M, I_1) \xrightarrow{\partial_1} \text{Hom}(M, I_2) \xrightarrow{\partial_2} \cdots$$

Now, $\text{Ext}^1(M, N) = \ker \partial_1/\text{im } \partial_0$, so $\Phi \in \ker \partial_1$. That is to say, $\partial_1(\Phi) = 0$. Now, recall that $\partial_1(\Phi) = f_1 \circ \Phi$, so we see that $f_1 \circ \Phi = 0$.

Since the injective resolution of N is exact, $\ker f_1 = \text{im } f_0 = I_0/N$; therefore, for any $m \in M$, $f_1 \circ \Phi(m) = 0$ means that $\Phi(m) \in$

$\ker f_1 = I_0/N$. Thus, if Φ is a representative of $\phi \in \mathrm{Ext}^1(M, N)$, $\Phi : M \to I_0/N$. $\square$

(b): Let $M' \to I_0$ be the pullback of $M \to I_0/N$ via the reduction map $I_0 \to I_0/N$. Show that the kernel of $M' \to M$ is N, giving an exact sequence $0 \to N \to M' \to M \to 0$, which corresponds to some class in $\mathrm{Ext}(M, N)$ that we denote by $c(\phi)$. (Here $\mathrm{Ext}(M, N)$ is the set of equivalence classes of extensions $0 \to N \to L \to M \to 0$ of M by N.)

Proof. We have the diagram:

$$\begin{array}{ccc} M' & \xrightarrow{\tilde{\pi}} & M \\ \downarrow{\scriptstyle \Phi'} & \square & \downarrow{\scriptstyle \Phi} \\ I_0 & \xrightarrow{\pi} & I_0/N \end{array}$$

By the definition of the pullback, M' is the fiber product $M \times_{I_0/N} I_0$. Hence, if $(m, i_0) \in \ker \tilde{\pi}$, then $0 = \pi'(m, i_0) = m$, so

$$\ker \tilde{\pi} = \{(0, i_0) \in M \times_{I_0/N} I_0 | \Phi(0) = \pi(i_0)\} \simeq \ker \pi$$

Now, since $\pi : I_0 \to I_0/N$ is just the standard projection, we see that $\ker \pi = N$, so we conclude that $\ker \tilde{\pi} = N$. Therefore, along the top row, we have the exact sequence

$$0 \longrightarrow N \longrightarrow M' \xrightarrow{\tilde{\pi}} M \longrightarrow 0,$$

which corresponds to some class in $\mathrm{Ext}(M, N)$ which we call $c(\phi)$. $\square$

(c): Show that $c : \mathrm{Ext}^1(M, N) \to \mathrm{Ext}(M, N)$ is a bijection.

Proof. To show that this is a bijection, we will define the inverse procedure. If

$$0 \longrightarrow N \xrightarrow{g} L \xrightarrow{h} M \longrightarrow 0 \tag{3}$$

is an extension of M by N, then, since I_0 is injective, $\mathrm{Hom}(\cdot, I_0)$ induces the exact sequence

$$0 \longrightarrow \mathrm{Hom}(M, I_0) \xrightarrow{h_*} \mathrm{Hom}(L, I_0) \xrightarrow{g_*} \mathrm{Hom}(N, I_0) \longrightarrow 0.$$

Now, $f_0 \in \text{Hom}(N, I_0)$, let $\widetilde{f_0} \in \text{Hom}(L, I_0)$ such that $f_0 = g_*(\widetilde{f_0}) = \widetilde{f_0} \circ g$. Then we let Φ be the push forward:

$$\begin{array}{ccc} L & \xrightarrow{h} & M \\ \big\downarrow{\scriptstyle \widetilde{f_0}} & \diamond & \big\downarrow{\scriptstyle \Phi} \\ I_0 & \xrightarrow{\pi} & I_0/N \end{array}$$

That is, let $\Phi(m) = \widetilde{f_0}(\ell)$ where $h(\ell) = m$. Note that this is well-defined because if $h(\ell') = h(\ell) = m$, then $\ell' = \ell + n$ for some $n \in \text{im } g = N$. Hence,

$$\widetilde{f_0}(\ell') = \widetilde{f_0}(\ell + n) = \widetilde{f_0}(\ell) + \widetilde{f_0}(n) = \widetilde{f_0}(\ell).$$

Now, since $I_0/N = \text{im } f_0 = \ker f_1$, we see that, in fact, $\partial_1(\Phi)(m) = f_1 \circ \Phi(m) = 0$, so $\Phi \in \ker \partial_1$, and so Φ is the representative of an element $\phi \in \text{Ext}^1(M, N)$. Hence the extension (3) determines an element $C(0 \to N \xrightarrow{g} L \xrightarrow{h} M \to 0) \in \text{Ext}^1(M, N)$. Now, if $\phi \in \text{Ext}^1(M, N)$ then Φ is a representative of ϕ and $c(\phi)$ is the extension

$$0 \longrightarrow N \xrightarrow{i} M \times_{I_0/N} I_0 \xrightarrow{\tilde{\pi}} M \longrightarrow 0$$

where the fiber product is relative to Φ. C applied to this extension yields Ψ where, for $m \in M$, if $\tilde{\pi}(\ell) = m$, then

$$\Phi'(\ell) = \Psi(m) = \Psi(\tilde{\pi}(\ell)) = \pi \circ \Phi'(\ell),$$

so $\Psi(m)$ and $\Phi(m)$ differ by an element of N and, thus, belong to the same equivalence class in $\text{Ext}^1(M, N)$, namely ϕ.

On the other hand, if

$$0 \longrightarrow N \xrightarrow{g} L \xrightarrow{h} M \longrightarrow 0$$

is an extension of M by N, then C applied to this extension yields the element of $\text{Ext}^1(M, N)$ γ represented by $\tilde{f_0}$; hence, $c(\gamma)$ is

$$0 \longrightarrow N \longrightarrow L \xrightarrow{\tilde{\pi}} M \longrightarrow 0$$

□

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