ALGEBRA HW 4

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1

(a): Show that if $0 \to M' \xrightarrow{f} M \xrightarrow{g} M'' \to 0$ is an exact sequence of R-modules, then M is Noetherian if and only if M' and M'' are.

Proof. ($\Rightarrow$) Suppose M is Noetherian. Then M' injects into M, so M' can be viewed as a submodule of M; since submodules of Noetherian modules are Noetherian, M' is Noetherian. Also, since the sequence is exact, $M'' \simeq M/f(M')$; since quotients of Noetherian modules are Noetherian, this implies that M'' is Noetherian.

($\Leftarrow$) On the other hand, suppose M' and M'' are Noetherian. Let $M_1 \subset M_2 \subset \cdots$ be a chain of submodules of M. Then, by problem set 3 #2, $f^{-1}(M_1) \subset f^{-1}(M_2) \subset \cdots$ is a chain of submodules of M' and $g(M_1) \subset g(M_2) \subset \cdots$ is a chain of submodules of M''. Since M' is Noetherian, eventually $f^{-1}(M_k) = f^{-1}(M_{k+1})$ for all $k \geq n_1$ for some $n_1 \in \mathbb{N}$. Similarly, since M'' is Noetherian, eventually $g(M_k) = g(M_{k+1})$ for all $k \geq n_2$ for some $n_2 = \mathbb{N}$.

Now, let $n = \max\{n_1, n_2\}$. Then, for all $k \geq n$, $f^{-1}(M_k) = f^{-1}(M_{k+1})$ and $g(M_k) = g(M_{k+1})$; using the result proved in problem set 3 #2(c), we have the following commutative diagram with exact rows for all $k \geq n$:

$$\begin{tikzcd}
0 \arrow[r] & f^{-1}(M_k) \arrow[r, "f"] \arrow[d, equal] & M_k \arrow[r, "g"] \arrow[d] & g(M_k) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & f^{-1}(M_{k+1}) \arrow[r, "f"] & M_{k+1} \arrow[r, "g"] & g(M_{k+1}) \arrow[r] & 0
\end{tikzcd}$$

Thus, by the five lemma, the map $M_k \to M_{k+1}$ is an isomorphism; since $M_k \subseteq M_{k+1}$ and this map is simply the inclusion map, this implies that $M_k = M_{k+1}$ for all $k \geq n$. Since our choice of a chain of submodules of M was arbitrary, we see that all chains of submodules of M eventually stabilize, which is precisely what it means for M to be Noetherian. $\square$

(b): Deduce that R-modules N_1 and N_2 are each Noetherian if and only if $N_1 \oplus N_2$ is.

Proof. For any R-modules N_1 and N_2, we have the natural exact sequence

$$0 \to N_1 \to N_1 \oplus N_2 \to N_2 \to 0;$$

by our result in part (a), this in turn means that N_1 and N_2 are each Noetherian if and only if $N_1 \oplus N_2$ is Noetherian. ☐

(c): Let S be a multiplicative subset of a commutative ring R. Show that if R is Noetherian then so is the localization $S^{-1}R$. Does the converse also hold?

Proof. Let $I = \left(\frac{r_1}{s_1}, \frac{r_2}{s_2}, \ldots\right)$ be an ideal in $S^{-1}R$. Let $a \in I$. Then

$$a = \sum_i b_i \frac{r_i}{s_i} = \sum_i \frac{b_i}{s_i} r_i$$

where all but finitely many b_i are zero. Then $\frac{b_i}{s_i} \in S^{-1}R$, so we see that a is a linear combination in $S^{-1}R$ of the r_i; since our choice of a was arbitrary, we see that, in fact, $I = (r_1, r_2, \ldots)$. Now, $(r_1, r_2, \ldots)$ can also be viewed as an ideal of R; since R is Noetherian, this ideal is finitely generated. That is, the contraction of I to R is equal to $(a_1, \ldots, a_n)$ for some $a_i \in R$. Consider $(a_1, \ldots, a_n)$ as an ideal in $S^{-1}R$. Certainly $(a_1, \ldots, a_n) \subset I$. To see the other containment, let $a \in I$. Then each $r_i = \sum_{i=1}^n c_i a_i$, so

$$a = \sum_i \frac{b_i}{s_i} r_i = \sum_i \frac{b_i}{s_i} \sum_{j=1}^n c_j a_j = \sum_{i,j} \frac{b_i c_j}{s_i} a_j.$$

Since all but finitely many of the b_i are zero, we see that a is a finite linear combination of the a_j and so $a \in (a_1, \ldots, a_n)$. Since our choice of a was arbitrary, we see that $I \subseteq (a_1, \ldots, a_n)$. Having shown both containments, we conclude that $I = (a_1, \ldots, a_n)$, so I is finitely generated. Since our choice of I was arbitrary, we conclude that all ideals of $S^{-1}R$ are finitely generated, and so $S^{-1}R$ is Noetherian.

The converse, on the other hand, fails to hold. To see why, consider $R = \mathbb{R}[x_1, x_2, \ldots]$ and let $S = \{x_1, x_2, \ldots\}$. Then, as we've seen before, $\mathbb{R}[x_1, x_2, \ldots]$ is not Noetherian (specifically, $(x_1, x_2, \ldots)$ is not finitely generated). However,

$$S^{-1}R = \mathbb{R}[x_1, x_2, \ldots]_{(x_1, x_2, \ldots)} = \mathbb{R}(x_1, x_2, \ldots)$$

is a field and so is trivially Noetherian. ☐

2

(a): Do there exist infinite strictly increasing chains $I_1 \subset I_2 \subset \cdots$ of ideals in $\mathbb{C}[x]$? Do there exist finite strictly increasing chains $I_1 \subset I_2 \subset \cdots \subset I_n$ of ideals in $\mathbb{C}[x]$ with arbitrarily large n?

Answer: Since $\mathbb{C}$ is Noetherian, $\mathbb{C}[x]$ is Noetherian as well, so, by the definition of what Noetherian means, there can be no infinite strictly increasing chains of ideals in $\mathbb{C}[x]$.

On the other hand, consider the following chain of strictly increasing ideals:

$$(x^n) \subsetneq (x^{n-1}) \subsetneq \cdots \subsetneq (x^2) \subsetneq (x) \subsetneq (1)$$

in $\mathbb{C}[x]$. We see that such a chain exists for all $n \in \mathbb{N}$, and so there are finite strictly increasing chains of ideals in $\mathbb{C}[x]$ of arbitrary length.

♣

(b): Repeat part (a), but with strictly *decreasing* chains of ideals $I_1 \supset I_2 \cdots$ r rather than strictly increasing chains.

Answer: Now that we're dealing with decreasing chains, consider the mirror image of the chain we just talked about, namely:

$$(1) \supsetneq (x) \supsetneq (x^2) \supsetneq \cdots .$$

This is an infinite strictly decreasing chain of ideals in $\mathbb{C}[x]$, so we see that such things exist. Furthermore, by stopping this chain at any point, we create a finite strictly decreasing chain of arbitrary length.

♣

(c): Explain geometrically your assertions in parts (a) and (b).

Answer: Associating each ideal in such a chain to the zero locus it corresponds to, essentially part (a) says that if we start with a finite collection of such loci, we can't take infinitely many away, although we can start with a finite collection that is as large as we want. On the other hand, if we start with some collection of spaces, we can keep adding more spaces (corresponding to smaller and smaller ideals) infinitely many times.

♣

3

Let M and N be finitely generated modules over a commutative ring R, such that $M \otimes_R N = 0$.

(a): Show that if R is a local ring with maximal ideal $\mathfrak{m}$, then M or N is 0.

Proof. Since $M \otimes_R N = 0$,

$$0 = (M \otimes_R N) \otimes_R \mathbb{R}/\mathfrak{m} = (M \otimes_R N) \otimes_R (R/\mathfrak{m} \otimes_{R/\mathfrak{m}} R/\mathfrak{m}) \simeq (M \otimes_R R/\mathfrak{m}) \otimes_{R/\mathfrak{m}} (N \otimes_R R/\mathfrak{m}).$$

Since $M \otimes_R R/\mathfrak{m} = M/\mathfrak{m}M$ and $N \otimes_R R/\mathfrak{m} = N/\mathfrak{m}N$, this implies that

$$0 = M/\mathfrak{m}M \otimes_{R/\mathfrak{m}} N/\mathfrak{m}N.$$

Now, since $\mathfrak{m}$ is a maximal ideal of R, $R/\mathfrak{m}$ is a field, so $M/\mathfrak{m}M$ and $N/\mathfrak{m}N$ are $R/\mathfrak{m}$-vector spaces. Now, if we take two non-zero vector fields V and W over a field k, then $V \otimes_k W \neq 0$ (specifically, if e_1 and f_1 are basis elements of V and W, then $e_1 \otimes f_1$ is part of a basis for $V \otimes_k W$). Hence, since $M/\mathfrak{m}M \otimes_{R/\mathfrak{m}} N/\mathfrak{m}N = 0$, this implies that either $M/\mathfrak{m}M = 0$ or $N/\mathfrak{m}N = 0$. Suppose that $M/\mathfrak{m}M = 0$. Then

$$0 = M/\mathfrak{m}M = M \otimes_R R/\mathfrak{m}$$

and so, by Nakayama's Lemma, $M = 0$. Similarly, if $N/\mathfrak{m}N = 0$, then $N = 0$

Therefore, we conclude that either $M = 0$ or $N = 0$. □

(b): What if R is not local?

Answer: The above statement does not necessarily hold if R is not local. For example, consider $R = \mathbb{Z}$ and the $\mathbb{Z}$-modules $\mathbb{Z}/2$ and $\mathbb{Z}/3$. Then $\mathbb{Z}/2 \otimes_{\mathbb{Z}} \mathbb{Z}/3 = 0$, even though neither $\mathbb{Z}/2$ nor $\mathbb{Z}/3$ is 0.

♣

4

Let M be an R-module and let $0 \to N' \to N \to N'' \to 0$ be an exact sequence of R-modules. Under each of the following four conditions (considered separately), either show that the sequence $0 \to M \otimes N' \to M \otimes N \to M \otimes N'' \to 0$ must be exact or else give a counterexample.

(a): M is flat.

Answer: By definition, if M is flat, then $0 \to M \otimes N' \to M \otimes N \to M \otimes N'' \to 0$ is exact.

♣

(b): N' is flat.

Counterexample: Consider the following exact sequence:

$$\begin{tikzcd} 0 \arrow[r] & \mathbb{Z} \arrow[r, "\cdot 2"] & \mathbb{Z} \arrow[r, "\pmod 2"] & \mathbb{Z}/2 \arrow[r] & 0. \end{tikzcd}$$

Let $M = \mathbb{Z}/2$. Then, since $\mathbb{Z}/2 \otimes_{\mathbb{Z}} \mathbb{Z} = \mathbb{Z}/2$ and $\mathbb{Z}/2 \otimes_{\mathbb{Z}} \mathbb{Z}/2 = \mathbb{Z}/2$, the application of $M \otimes_{\mathbb{Z}} \cdot$ yields the following sequence:

$$\begin{tikzcd} 0 \arrow[r] & \mathbb{Z}/2 \arrow[r, "\cdot 2"] & \mathbb{Z}/2 \arrow[r, "id"] & \mathbb{Z}/2 \arrow[r] & 0. \end{tikzcd}$$

Now, in this sequence, multiplying by 2 kills everything in $\mathbb{Z}/2$, so $\ker \cdot 2 = \mathbb{Z}/2 \neq 0 = \operatorname{im} 0$, so this sequence is not exact.

♣

(c): N is flat.

Counterexample: In the example given in (b) above, the middle term, $\mathbb{Z}$, is flat, so the example in (b) is also a counterexample for this situation.

♣

(d): N'' is flat.

Answer: By the definition of $\operatorname{Tor}^*(M,\cdot)$ as the universal δ-functor for $M \otimes \cdot$,

$$\begin{tikzcd}\cdots \arrow[r] & \operatorname{Tor}^1(M, N'') \arrow[r] & M \otimes N' \arrow[r] & M \otimes N \arrow[r] & M \otimes N'' \arrow[r] & 0\end{tikzcd}$$

is a long exact sequence. Now, since N'' is flat and $\operatorname{Tor}^*(\cdot, N)$ is also the universal δ-functor for $\cdot \otimes N$ (that is, Tor is a balanced functor), it must be the case that $\operatorname{Tor}^1(M, N'') = 0$; hence, the tail end of the above sequence reduces to

$$\begin{tikzcd}0 \arrow[r] & M \otimes N' \arrow[r] & M \otimes N \arrow[r] & M \otimes N'' \arrow[r] & 0\end{tikzcd}$$

which must also be exact.

♣

5

Let R be a commutative ring with Jacobson radical J. For $x \in R$, let P_x be the set of $r \in R$ such that $r \equiv 1 \pmod{x}$ (i.e. such that $r - 1 \in xR$). Let R^* denote the multiplicative group of units in R.

(a): Show that $J = \{x \in R | P_x \subset R^*\}$.

Proof. Let $S = \{x \in R | P_x \subset R^*\}$. Let $x \in S$. Let $\mathfrak{m}$ be a maximal ideal of R and suppose $x \notin \mathfrak{m}$. Since $\mathfrak{m}$ is maximal, this implies that $(x, \mathfrak{m}) = (1)$, so there exists $a \in R$ and $b \in \mathfrak{m}$ such that $1 = ax + b$. Re-arranging this equation yields:

$$b - 1 = -ax \in xR,$$

so $b \in P_x \subset R^*$. However, this implies that b is a unit and so that $\mathfrak{m} = (1)$, contradicting the fact that $\mathfrak{m}$ is maximal and, therefore, proper.

From this contradiction, then, we see that $x \in \mathfrak{m}$. Since our choice of maximal ideal $\mathfrak{m}$ was arbitrary, this in turn implies that $x \in \mathfrak{m}$ for all maximal ideals $\mathfrak{m}$ and, hence,

$$x \in \bigcap_{\mathfrak{m} \text{ maximal}} \mathfrak{m} = J.$$

Since our choice of $x \in S$ was arbitrary, this implies that $S \subseteq J$.

On the other hand, suppose $x \in J$. Let $r \in P_x$. Then $r - 1 \in xR \subseteq J$. If r is not invertible, then (r) is a proper ideal and so $r \in \mathfrak{m}$ for some maximal ideal $\mathfrak{m}$. However, since $r - 1 \in J$, $r - 1 \in \mathfrak{n}$ for all maximal $\mathfrak{n}$; specifically, $r - 1 \in \mathfrak{m}$. Hence, $r, r - 1 \in \mathfrak{m}$ and so $1 = r - (r - 1) \in \mathfrak{m}$, meaning $\mathfrak{m} = (1)$, contradicting the fact that $\mathfrak{m}$ is maximal and, therefore, proper. Hence, we see that $r \in R^*$. Since our choice of $r \in P_x$ was arbitrary, we see that $P_x \subseteq R^*$ and so $x \in S$. Since our choice of x was arbitrary, this in turn implies that $J \subseteq S$. Thus, having shown containment in both directions, we conclude that $J = S$. □

(b): Let M be a finitely generated R-module and let $a_1, \dots, a_n \in M$. Show that the R-module M is generated by $a_1, \dots, a_n$ if and only if the R/J-module M/JM is generated by the images of these elements.

Proof. Certainly, if $a_1, \dots, a_n$ generate M, then their images must generate M/JM. To prove the converse, we will need the following generalization of Nakayama's Lemma:

Lemma 5.1. *Let R be a ring with Jacobson radical J and M a finitely generated R-module. If $M \otimes R/J = 0$, then $M = 0$.*

Proof. Suppose $M \neq 0$. Let $b_1, \dots, b_k$ be a minimal generating set of M. Then $k \geq 1$. Since $0 = M \otimes R/J = M/JM$, we have that $M = JM$. Then $b_1 \in M$ and

$$b_1 = \sum_{i=1}^{k} j_i b_i$$

for $j_i \in J$. Hence,

$$(1 - j_1)b_1 = \sum_{i=2}^{k} j_i b_i.$$

Now, $(1 - j_1) - 1 = -j_1 \in J$, so, by the result proved in (a), $1 - j_1 \in R^*$. Let $h = (1 - j_1)^{-1}$. Then multiplying both sides above by h, we see that

$$b_1 = \sum_{i=2}^{k} h j_i b_i.$$

Thus, b_1 is a linear combination of $b_2, \dots, b_k$, so $\langle b_1, \dots, b_k \rangle = \langle b_2, \dots, b_k \rangle$, meaning that $\{b_1, \dots, b_k\}$ was not a minimal generating set. From this contradiction, then, we conclude that, in fact, $M = 0$. □

Now, suppose the images $\overline{a}_1, \dots, \overline{a}_n$ generate M/JM. Let N be the submodule of M generated by $a_1, \dots, a_n$. Then

$$M/N \otimes_R R/J = 0,$$

so, by Lemma 5.1, $M/N = 0$. Therefore, $N = M$, so we see that $a_1, \dots, a_n$ generate M. □

6

Let M be a finitely generated R-module. Prove that the following conditions on M are equivalent:

(i): M is locally free over R (i.e. $M_{\mathfrak{m}}$ is free over $R_{\mathfrak{m}}$ for all maximal ideals $\mathfrak{m} \subset R$).

(ii): For every maximal ideal $\mathfrak{m} \subset R$ there is an $f \notin \mathfrak{m}$ such that M_f is free over R_f.

(iii): $\exists f_1, \ldots, f_n \in R$ such that $(f_1, \ldots, f_n) = 1$ and M_{f_i} is free over R_{f_i} for $i = 1, \ldots, n$.

Proof. (i) $\Rightarrow$ (ii): Suppose $\mathfrak{m}$ is a maximal ideal of R. Then, by hypothesis, $M_{\mathfrak{m}}$ is free over $R_{\mathfrak{m}}$. Let $m_1, \ldots, m_n$ be a basis for $M_{\mathfrak{m}}$ over $R_{\mathfrak{m}}$ and let $\{g_1, \ldots, g_n\}$ be a generating set for M. Then for each g_i,

$$g_i = \sum_{j=1}^{n} \frac{r_{i,j}}{s_{i,j}} m_j$$

for $r_{i,j} \in R$ and $s_{i,j} \notin \mathfrak{m}$. Now, let

$$f = \prod_{i,j} s_{i,j}.$$

Note that $f \notin \mathfrak{m}$. Let $\frac{h}{f^N} \in M_f$. Then $h \in M$ and so $h = \sum_{i=1}^{n} a_i g_i$ for some $a_i \in R$. Hence,

$$\frac{h}{f^N} = \frac{\sum_{i=1}^{n} a_i g_i}{f^N} = \sum_i \frac{a_i}{f^N} \sum_j \frac{r_{i,j}}{s_{i,j}} m_j.$$

However, since

$$\frac{1}{s_{i,j}} = \frac{f/s_{i,j}}{f}$$

and, by construction, $f/s_{i,j} \in R$, we see that

$$\frac{h}{f^N} = \sum_{i,j} \frac{a_i r_{i,j} (f/s_{i,j})}{f^{N+1}} m_j.$$

Note that $\frac{a_i r_{i,j}(f/s_{i,j})}{f^{N+1}} \in M_f$. Thus the m_j form a generating set for M_f. Since the m_j are linearly independent in $M_{\mathfrak{m}} \supset M_f$, we see that $\{m_1, \ldots, m_n\}$ forms a basis for M_f over R_f, so M_f is free over R_f.

(ii) $\Rightarrow$ (iii): Let $S = \{f \in R | M_f \text{ is free over } R_f\}$. Let (S) be the ideal generated by the elements of S, and suppose $(S) \neq (1)$. Then, since (S) is a proper ideal of R, there exists a maximal ideal $\mathfrak{m}$ containing (S). Then, by hypothesis, there exists $f \notin \mathfrak{m}$ such that M_f is free over R_f. However, this implies that $f \in S$ and, hence, $f \in (S) \subset \mathfrak{m}$. From this contradiction, then, we see that $(S) = (1)$. Since M is finitely generated, there is some finite generating set $f_1, \ldots, f_k \in S$ such that $(f_1, \ldots, f_k) = (S) = (1)$. Since each $f_i \in S$, M_{f_i} is free over R_{f_i} for all $i = 1, \ldots, n$.

(iii) $\Rightarrow$ (i): Let $\mathfrak{m}$ be a maximal ideal of R. By hypothesis, there exist $f_1, \ldots, f_n \in R$ such that $(f_1, \ldots, f_n) = (1)$ and M_{f_i} is free over R_{f_i} for all i. Now, since $\mathfrak{m}$ is proper, there exists some f_k such that $f_k \notin \mathfrak{m}$ and M_{f_k} is free over R_{f_k}. Since free implies locally free and $\mathfrak{m}_{f_k}$ is maximal in R_{f_k}, this in turn implies that $(M_{f_k})_{\mathfrak{m}_{f_k}}$ is free over $(R_{f_k})_{\mathfrak{m}_{f_k}}$. However, since $f_k \in R\backslash\mathfrak{m}$, $(M_{f_k})_{\mathfrak{m}_{f_k}} = M_\mathfrak{m}$ and $(R_{f_k})_{\mathfrak{m}_{f_k}} = R_\mathfrak{m}$, so we see that $M_\mathfrak{m}$ is free over $R_\mathfrak{m}$. □

7

Let M be an R-module. Prove that the following conditions on M are equivalent:

(i): M is faithfully flat.

(ii): For every *complex* of R-modules $N' \to N \to N''$, the given complex is exact iff $M \otimes N' \to M \otimes N \to M \otimes N''$ is exact.

(iii): For every homomorphism of R-modules $\phi : N_1 \to N_2$, ϕ is surjective iff $1 \otimes \phi : M \otimes N_1 \to M \otimes N_2$ is surjective.

Proof. (i) ⇒ (ii): If $N' \to N \to N''$ is a complex, then it is also a sequence and so, since M is faithfully flat, $N' \to N \to N''$ is exact if and only if $M \otimes N' \to M \otimes N \to M \otimes N''$ is exact.

(ii) ⇒ (iii): Let $\phi : N_1 \to N_2$ be surjective. Then $N_1 \xrightarrow{\phi} N_2 \to 0$ is exact, and so, by (ii), $M \otimes N_1 \xrightarrow{1\otimes\phi} M \otimes N_2 \to 0$ is exact and, therefore, $1 \otimes \phi : M \otimes N_1 \to M \otimes N_2$ is surjective. On the other hand, if $1 \otimes \phi : M \otimes N_1 \to M \otimes N_2$ is surjective, then $M \otimes N_1 \xrightarrow{1\otimes\phi} M \otimes N_2 \to 0$ is exact; hence, by (ii), $N_1 \xrightarrow{\phi} N_2 \to 0$ is exact, meaning that $\phi : N_1 \to N_2$ is surjective.

(iii) ⇒ (i): □

8

Let A be a flat R-algebra. Prove that the following conditions on A are equivalent:

(i): A is faithfully flat over R.

(ii): Every maximal ideal of R is the contraction of a maximal ideal of A.

(iii): Spec $A \to$ Spec R is surjective.

(iv): For every R-module N, if $A \otimes_R N = 0$, then $N = 0$.

Proof. (i) ⇒ (ii) Let $\mathfrak{m}$ be a maximal ideal in R. Then $\mathfrak{m} \hookrightarrow R$, the inclusion map, is not surjective. Therefore, by the contrapositive of part of problem 7 above,

$$\mathfrak{m} \otimes_R A \hookrightarrow R \otimes_R A$$

is not a surjection. Now, $\mathfrak{m} \otimes_R A = \mathfrak{m}^e$, the extension of $\mathfrak{m}$ to A. On the other hand, $R \otimes_R A = A$, so we see that the inclusion $\mathfrak{m}^e \hookrightarrow A$ is not a surjection, so $\mathfrak{m}^e$ is a proper ideal of A and, therefore, is contained in some maximal ideal $\mathfrak{n}$ of A. Now, if we denote the contraction of an ideal $\mathfrak{a}$ of A to R by $\mathfrak{a}^c$, then

$$\mathfrak{m} \subseteq \mathfrak{m}^{ec} \subseteq \mathfrak{n}^c.$$

However, since $\mathfrak{n}$ is a proper ideal, $1 \notin \mathfrak{n}$ and so $1 \notin \mathfrak{n}^c$, meaning $\mathfrak{n}^c$ is a proper ideal of R. Therefore, since $\mathfrak{m}$ is maximal in R, it must be the case that $\mathfrak{m} = \mathfrak{n}^c$, so $\mathfrak{m}$ is the contraction of a maximal ideal of A. Since our choice of maximal ideal $\mathfrak{m}$ was arbitrary, we conclude that every maximal ideal of R is the contraction of a maximal ideal of A.

(ii) $\Rightarrow$ (iv) Suppose $N \neq 0$ is an R-module such that $N \otimes_R A = 0$. Let $x \neq 0$ be an element of N. Then the map $f : R \to xR$ given by

$$a \mapsto xa$$

is surjective. Hence, if $\mathfrak{k} = \ker f$, $xR \simeq R/\mathfrak{k}$. Now, consider the module

$$xR \otimes_R A \simeq R/\mathfrak{k} \otimes_R A \simeq A/\mathfrak{k}^e,$$

where $\mathfrak{k}^e$ denotes the extension of the ideal $\mathfrak{k}$ to A.

Now, since $f(1) = x \neq 0$, $\mathfrak{k}$ is a proper ideal of R and so is contained in some maximal ideal $\mathfrak{m}$. By hypothesis, $\mathfrak{m} = \mathfrak{n}^c$ for some maximal $\mathfrak{n}$ in A. Therefore,

$$A \supsetneq \mathfrak{n} \supset \mathfrak{k}^e,$$

so $A/\mathfrak{k}^e \neq 0$ and, hence, $xR \otimes_R A \neq 0$. Now, since A is flat and $0 \to xR \to R$ given by the inclusion map is exact,

$$0 \to xR \otimes_R A \to R \otimes_R A$$

is exact; specifically, the map $xR \otimes A \to R \otimes A$ is injective. Now, as we've just seen, $xR \otimes A \neq 0$, so this implies that $R \otimes A \neq 0$. From this contradiction, then, we conclude that for every R-module N, if $N \otimes_R A = 0$, then $N = 0$.

(iv) $\Rightarrow$ (i) Suppose $\phi : N_1 \to N_2$ such that $1 \otimes \phi : A \otimes N_1 \to A \otimes N_2$ is surjective. Then

$$0 \to \phi(N_1) \to N_2 \to N_2/\phi(N_1) \to 0$$

is exact. Since A is flat,

$$0 \to A \otimes \phi(N_1) \to A \otimes N_2 \to A \otimes (N_2/\phi(N_1)) \to 0$$

is exact. Now, we can distribute the tensor across quotients, so $A \otimes (N_2/\phi(N_1)) \simeq (A \otimes N_2)/(A \otimes \phi(N_1)$.

Recall that $1 \otimes \phi : A \otimes N_1 \to A \otimes N_2$ is surjective. Thus,

$$A \otimes \phi(N_1) = \text{im } \phi = A \otimes N_2,$$

so $0 = (A \otimes N_2)/(A \otimes \phi(N_1)) = A \otimes (N_2/\phi(N_1))$. Therefore, by hypothesis, $N_2/\phi(N_1) = 0$, and so $N_2 = \phi(N_1)$, meaning that $\phi : N_1 \to N_2$ is surjective.

(i) $\Rightarrow$ (iii) Since we've shown that (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iv), we should feel free to use (ii) or (iv) if they should come in handy. Now, first of all, let $\mathfrak{p}$ be a prime ideal of R and consider the localization $A_{\mathfrak{p}}$ as a module over $R_{\mathfrak{p}}$. We suppose first that $A_{\mathfrak{p}}$ is faithfully flat over

$R_{\mathfrak{p}}$ (which we'll show in a minute). Note that $\mathfrak{p}R_{\mathfrak{p}}$ is the unique maximal ideal in $\mathfrak{p}$; by (ii) there exists a maximal $\mathfrak{m}$ in $A_{\mathfrak{p}}$ such that $\mathfrak{m}^c = \mathfrak{p}R_{\mathfrak{p}}$. Consider the following diagram:

$$\begin{tikzcd}
R \arrow[r, "\phi"] \arrow[d, leftrightarrow] & A \arrow[d, leftrightarrow] \\
R_{\mathfrak{p}} \arrow[r, "\phi'"] & A_{\mathfrak{p}}
\end{tikzcd}$$

where ϕ' is the map induced by the map $\phi : R \to A$. Then $\mathfrak{p}R_{\mathfrak{p}} = \mathfrak{m}^c = \phi'^{-1}(\mathfrak{m})$. Let $\mathfrak{q} = A \cap \mathfrak{m}$. Then

$$\phi^{-1}(\mathfrak{q}) = \phi^{-1}(\mathfrak{m} \cap A) = \phi^{-1}(\mathfrak{m}) \cap \phi^{-1}(A) = \mathfrak{p}.$$

Since our choice of $\mathfrak{p}$ was arbitrary, we see that Spec $A \to$ Spec R is surjective.

Now, to justify our assumption that $A_{\mathfrak{p}}$ is faithfully flat over $R_{\mathfrak{p}}$, suppose $\mathfrak{p} \in$ Spec R. Then $R_{\mathfrak{p}}$ is flat over R and $A_{\mathfrak{p}} = A \otimes_R R_{\mathfrak{p}}$. Now, suppose M and N are $R_{\mathfrak{p}}$ modules such that $f : M \to N$ induces a surjective map $1 \otimes f : A_{\mathfrak{p}} \otimes_{R_{\mathfrak{p}}} M \to A_{\mathfrak{p}} \otimes_{R_{\mathfrak{p}}} N$. Then, since $A_{\mathfrak{p}} = A \otimes_R R_{\mathfrak{p}}$, the following sequence is exact:

$$(A \otimes_R R_{\mathfrak{p}}) \otimes_{R_{\mathfrak{p}}} M \xrightarrow{1 \otimes f} (A \otimes_R R_{\mathfrak{p}}) \otimes_{R_{\mathfrak{p}}} N \longrightarrow 0.$$

Now $(A \otimes_R R_{\mathfrak{p}}) \otimes_{R_{\mathfrak{p}}} M = A \otimes_R (R_{\mathfrak{p}} \otimes_{R_{\mathfrak{p}}} M) = A \otimes_R M$ and similarly for N, so the above sequence reduces to:

$$A \otimes_R M \to A \otimes_R N \to 0.$$

Since A is faithfully flat over R, this in turn implies that $M \to N \to 0$ is exact, which is to say that $f : M \to N$ is surjective.

(iii) $\Rightarrow$ (ii) Suppose $\mathfrak{m}$ is maximal in R. Then $\mathfrak{m}$ is prime in R, so $\mathfrak{m} = \mathfrak{p}^c$ for some prime ideal $\mathfrak{p}$ of A by hypothesis. Now, $\mathfrak{p}$ is contained in some maximal ideal $\mathfrak{n}$; since $\mathfrak{n}$ is proper, it does not contain 1, so $\mathfrak{n}^c$ does not contain 1. Furthermore, since $\mathfrak{m} = \mathfrak{p}^c \subset \mathfrak{n}^c$, we see that, in fact, $\mathfrak{m} = \mathfrak{n}^c$. Since our choice of $\mathfrak{m}$ was arbitrary, we conclude that every maximal ideal of R is the contraction of a maximal ideal of A. □

9

For each of the following R-algebras A, determine whether A is a finitely generated R-module and whether it is a finitely generated R-algebra. Also determine whether the R-module A is flat and whether it is faithfully flat.

(a): $R = \mathbb{Z}$, $A = \mathbb{Z}[x]/(3x)$.

Answer: A is not finitely generated as an R-module. To see why, suppose $\{f_1, \ldots, f_n\}$ were a finite generating set. Let $N =$

$\max\{\deg(f_i)\}$. Then for any $a_i \in R$,

$$\sum_{i=1}^{n} a_i f_i$$

has degree at most N, so, for example, x^{N+1} cannot be written as such a linear combination. Since $x^{N+1} \in A$, this implies that A is not finitely generated as an R-module.

On the other hand, $\{1, x\}$ serves as a generating set for A as an R-algebra, so A is finitely-generated as an R-algebra. Now, since $x \in A$ and $3x = 0$ in A, we see that A has 3-torsion elements; since a flat module over a domain cannot have torsion, this implies that A is not a flat module.

♣

(b): $R = \mathbb{Z}$, $A = \mathbb{Z}[1/5]$.

Answer: A is not finitely generated as an R-module. To see why, suppose $\{a_1, \ldots, a_n\}$ were a finite generating set. Then each

$$a_i = \sum_{j=1}^{k_i} \frac{b_{i,j}}{5^{l_{i,j}}}.$$

Let $N = \max\{l_{i,j}\}$. Then $\frac{1}{5^{N+1}}$ cannot be written as a linear combination of the a_i, so we see that A is not finitely-generated as an R-module.

On the other hand, $\left\{1, \frac{1}{5}\right\}$ does serve as a generating set of A as an R-algebra, so we see that A is finitely generated as an R-algebra. Furthermore, since A is an integral domain and is a ring extension of R, A is torsion-free as an R-module. Therefore, since torsion-free modules over a PID are flat and $\mathbb{Z}$ is a PID, we see that A is a flat module.

Now, consider the maximal ideal (5) in $\mathbb{Z}$. 5 is a unit in $A = \mathbb{Z}[1/5]$, so we see that 5 cannot be contained in any ideals of A. Hence, (5) is not the contraction of any maximal ideal in A and so, by the criterion for faithful flatness given in #8(ii), A is not faithfully flat.

♣

(c): $R = \mathbb{Z}$, $A = \mathbb{Z}[i]$.

Answer: Note that any element of the Gaussian integers is of the form $a + bi$ for $a, b \in \mathbb{Z}$. Hence, we see that $\{1, i\}$ forms a generating set for A as an R-algebra, so A is finitely-generated as an R-module (and, consequently, also as an R-algebra). Furthermore, if $a + bi = 0$ for some $a, b \in \mathbb{Z}$, then $a = b = 0$, so we see that, in fact, A is a free R-module and, therefore, flat. Since $\{1, i\}$ is a basis for A, $A \simeq R^2$ and so, if N is an R-module such that $0 = A \otimes_R N$, then

$$0 = A \otimes_R N \simeq R^2 \otimes_R N = (R \otimes_R N) \oplus (R \otimes_R N) = N^2,$$

so $N = 0$. Hence, by criterion 8(iv), A is faithfully flat.

♣

(d): $R = \mathbb{Z}$, $A = \mathbb{Z}[i, 1/5]$.

Answer: Since we saw in (b) that $\mathbb{Z}[1/5]$ is not finitely generated as a $\mathbb{Z}$-module and since $A = \mathbb{Z}[i, 1/5] \simeq \mathbb{Z}[1/5][i]$, this implies that A is not finitely generated as a $\mathbb{Z}$-module. However, it's clear that $\left\{1, \frac{1}{5}, i\right\}$ form a generating set for A as an R-algebra, so A is finitely generated as an R-algebra. Furthermore, since A is an integral domain and a ring extension of R, A is torsion-free and, therefore, flat. However, (5) is a maximal ideal in $\mathbb{Z}$, whereas 5 is a unit in $\mathbb{Z}[i, 1/5]$, so 5 is not contained in any ideals of A and, therefore, (5) is not the contraction of any maximal ideal of A. Therefore, by criterion 8(ii), A is not faithfully flat.

♣

(e): $R = \mathbb{Z}$, $A = \mathbb{Z}[i, 1/(2+i)]$.

Answer: A is not finitely-generated as an R-algebra. To see why, suppose $\{a_1, \ldots, a_n\}$ were a finite generating set. Then each

$$a_j = \sum_{k=1}^{l_j} \frac{b_{j,k} i^{j,k}}{(2+i)^{m_{j,k}}}.$$

Let $N = \max\{m_{j,k}\}$. Then $\frac{1}{(2+i)^{N+1}}$ cannot be written as a linear combination of the a_j, so we see that A is not finitely-generated as an R-module.

On the other hand, $\left\{1, i, \frac{1}{2+i}\right\}$ is a generating set for A an an R-algebra, so A is finitely-generated as an R-algebra. Furthermore, since A is an integral domain and a ring extension of R, A is torsion-free and, therefore, flat.

Finally, note that the maximal ideals in R are of the form (p) for p prime. Then, as we saw on PS8#r(a) from last semester, either p is prime in $\mathbb{Z}[i]$ or $p = \beta_p \overline{\beta}_p$ for β_p and $\overline{\beta}_p$ prime in $\mathbb{Z}[i]$. So long as $p \neq 5$, $\beta_p, \overline{\beta}_p \neq 2 + i$, so $\beta_p, \overline{\beta}_p$ are prime in A. Furthermore, as we saw in PS9#4(c), the contraction of (β_p) to $\mathbb{Z}$ is just (p), so we see that (p) is the contraction of a maximal ideal in A to R. Of course, $5 = (2+i)(2-i)$; since $2+i$ is a unit in A, we have to be a little more careful. However, $\frac{1}{2-i} \notin A$ and so $2 - i$ is not a unit in A. In addition, $2 - i$ is prime in $\mathbb{Z}[i]$ and is not divisible by $\frac{1}{2+i}$ or any of its multiples, so $2 - i$ is prime in A. Furthermore, the contraction of $\left(\frac{1}{2-i}\right)$ is, by the same reasoning, simply (5). Therefore, we see that all maximal ideals of $\mathbb{Z}$ are contractions of maximal ideals in A and so, by criterion 8(ii), A is faithfully flat.

♣

(f): $R = \mathbb{R}[x]$, $A = \mathbb{R}[[x]]$.

Answer: A is not finitely-generated as an R-module. To see why, suppose $\{f_1(x), \ldots, f_n(x)\}$ were a finite generating set for A. Let

$$h(x) = \sum_{i=1}^{n} g_i(x) f_i(x)$$

be a linear combination of the f_i. Then h is simply a polynomial with real coefficients. However, there are elements of A that are not polynomials, so we see that this cannot be a generating set for A and, hence, A is not finitely-generated as an R-module.

Since $\mathbb{R}[x]$ is a PID and A is torsion-free, A is a flat R-module. However, A is local and so has exactly one maximal ideal, whereas R has infinitely many maximal ideals (for example, $(x-a)$ is maximal for all $a \in \mathbb{R}$), so we see that not every maximal ideal in R can be the contraction of a maximal ideal in A, so, by criterion 8(ii), A is not faithfully flat.

♣

(g): $R = \mathbb{R}[x]_{(x)}$, $A = \mathbb{R}[[x]]$.
Answer:

♣

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