ALGEBRA HW 5

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1

(a): Let V be an affine variety, with ring of functions R. Let W be a Zariski closed subset of V, and let $I = I(W)$. Show that W is irreducible if and only if I is a prime ideal.

Proof. ($\Rightarrow$) By contrapositive. Suppose, first of all, that I is not prime. Then there exist $f, g \in R$ such that $fg \in I$, but $f \notin I$, $g \notin I$. If $V(f) \supset W$, then f vanishes on W; since $I = I(W)$, this means that $f \in I$. Similarly, if $V(g) \supset W$, then g vanishes on W and so $g \in I$. Hence, we see that $V(f) \not\supset W$ and $V(g) \not\supset W$ and, hence, $W \neq V(f) \cap W$ and $W \neq V(g) \cap$. Now, if $p \in W$, then fg vanishes at p, meaning that either f or g must vanish at p. In turn, this implies that either $p \in V(f)$ or $p \in V(g)$. Hence, $p \in V(f) \cup V(g)$. Since our choice of p was arbitrary, we see that $W \subset V(f) \cup V(g)$, and so $W = (V(f) \cup V(g)) \cap W = (V(f) \cap W) \cup (V(g) \cap W)$. Since neither term in this union comprises all of W, we see that W is reducible.

($\Leftarrow$) Again by contrapositive. Suppose W is reducible; that is, $W = W_1 \cup W_2$ for $W_1 \neq W$, $W_2 \neq W$ where W_1 and W_2 are subvarieties. Then $W_1 = V(I_1)$ and $W_2 = V(I_2)$ for ideals $I_1, I_2 \subset R$. Since neither of W_1 nor W_2 is all of W, this means that there must exist $f \in I_1$ and $g \in I_2$ such that f doesn't vanish on all of W_2 and g doesn't vanish on all of W_1; that is $f \notin I$ and $g \notin I$. Now, since f vanishes on W_1 and g vanishes on W_2, fg must vanish on all of $W_1 \cup W_2 = W$, and so $fg \in I$. Since $f \notin I$ and $g \notin I$, this implies that I is not prime. □

(b): Let $I_1, \ldots, I_n$ be proper ideals of R, and let $W_i = V(I_i)$ for each i. Show that

$$V(I_1 + \cdots + I_n) = W_1 \cap \cdots \cap W_n,$$
$$V(I_1 \cap \cdots \cap I_n) = W_1 \cup \cdots \cup W_n.$$

Also explain the relationship with problem 8 on math 602 Problem Set 8.

Proof. If $a \in V(I_1 + \cdots + I_n)$, then for all $f_i \in I_i$, f_i vanishes at p. Hence, $p \in W_i$ for all $i = 1, \ldots, n$ and, hence, $p \in W_1 \cap \cdots \cap W_n$. Since our choice of p was arbitrary, we see that $V(I_1 + \cdots + I_n) \subset W_1 \cap \cdots \cap W_n$. On the other hand, if $p \in W_1 \cap \cdots \cap W_n$, then for any $i = 1, \dots, n$ and any $f_i \in I_i$, f_i vanishes at p. Therefore, a generic element $\sum_{i=1}^n f_i \in I_1 + \ldots + I_n$ must vanish at p, and so $p \in V(I_1 + \cdots + I_n)$. Since our choice of p was arbitrary, we conclude that $W_1 \cap \cdots \cap W_n \subset V(I_1 + \cdots + I_n)$; having shown both containments, we conclude that $V(I_1 + \cdots + I_n) = W_1 \cap \cdots \cap W_n$.

Turning to the second equality, suppose $p \in V(I_1 \cap \cdots \cap I_n)$. Then for $f_i \in I_i$, f_i vanishes at p for all $i = 1, \dots, n$. Hence, $p \in W_i$ for all i and so certainly $p \in W_1 \cup \cdots \cup W_n$. SInce our choice of p was arbitrary, we see that $V(I_1 \cap \cdots \cap I_n) \subset W_1 \cup \cdots \cup W_n$. On the other hand, if $p \in W_1 \cup \cdots \cup W_n$, then $p \in W_i$ for some i. Hence, $f \in I_1 \cap \cdots \cap I_n \subset I_i$ must vanish at p since any element of I_i vanishes on all of W_i. Since our choice of p was arbitrary, we see that $W_1 \cup \cdots \cup W_n \subset V(I_1 \cap \cdots \cap I_n)$; having shown containment in both directions, we conclude that $V(I_1 \cap \cdots \cap I_n) = W_1 \cup \cdots \cup W_n$. □

2

(a): Let R be a Noetherian ring and $I \subset R$ an ideal. Prove that there are only finitely many prime ideals that are minimal over I.

Proof. Suppose, to the contrary, that there are infinitely many prime ideals minimal over I. Let S be the set of ideals having this property (i.e. S is the set of ideals in R with infinitely many primes minimal over them). Then S is partially ordered by containment. If $I_1 \subsetneq I_2 \subsetneq \cdots$ is a strictly ascending chain in S then, since R is Noetherian, this chain must be finite. Let I_n be the maximal element of this chain; that is, there is no ideal J containing I_n such that J has infinitely many primes minimal over it. Now, clearly I_n is not prime, else it would be the unique prime ideal minimal over itself. Therefore, there exist $a, b \in R - I_n$ such that $ab \in I_n$. Now, if $\mathfrak{p}$ is a prime ideal minimal over I, then, since $ab \in \mathfrak{p}$, $a \in \mathfrak{p}$ or $b \in \mathfrak{p}$. Suppose, without loss of generality, that $a \in \mathfrak{p}$. Clearly, $\mathfrak{p} \supset I_n + (a)$. Furthermore, if $\mathfrak{q}$ is another prime ideal containing $I_n + (a)$, then $I_n \subset \mathfrak{q}$ and so $\mathfrak{q} \supset \mathfrak{p}$ since $\mathfrak{p}$ is minimal over I_n. Therefore, $\mathfrak{p}$ is minimal over the ideal $I_n + (a)$. Since our choice of $\mathfrak{p}$ was arbitrary, we see that all of the infinitely many prime ideals minimal over I_n are minimal over either $I_n + (a)$ or $I_n + (b)$. Hence, one of these must have infinitely many primes minimal over it; again, suppose it is $I_n + (a)$. However, $I_n + (a) \supset I_n$, contradicting the fact that I_n is maximal.

Therefore, we conclude that only finitely many primes are minimal over I. □

(b): Deduce that every Noetherian ring has finitely many minimal primes. Also, interpret this assertion geometrically, if R is the ring of functions on a Zariski closed subset of an affine variety V.

Proof. If R is a Noetherian ring, then (0) is an ideal of R. Therefore, by the result proved in (a), (0), has only finitely many prime ideals minimal over it. However, the primes minimal over (0) are precisely the minimal primes in R, so we see that R has only finitely many minimal primes. Geometrically, if R is the ring of functions on a Zariski closed subset W of an affine variety V, then the minimal primes correspond to the connected components of W, so we see that W has only finitely many connected components. □

(c): What happens in (a) and (b) if the ring is not Noetherian? If $R = \prod_{i=1}^{\infty} \mathbb{C}$, then the ideal generated by $(0, 0, \ldots)$ is just the zero ideal, which is not prime in R, since $(1, 0, 0, \ldots) \cdot (0, 1, 0, 0, \ldots) = (0, 0, 0, \ldots)$. Now, the ideal I_k generated by $(0, 0, \ldots, 0, 1, 0, \ldots)$ is minimal over the zero ideal (and, hence, minimal in R), since any ideal contained in I_k would have to contain some non-zero $(0, 0, \ldots, 0, a, 0, \ldots)$ but not $(0, 0, \ldots, 0, 1, 0, \ldots)$; however,

$$(0, 0, \ldots, 0, a, 0, \ldots) \cdot (0, 0, \ldots, 0, a^{-1}, 0, \ldots) = (0, 0, \ldots, 0, 1, 0, \ldots),$$

so this is impossible. Furthermore, I_k is prime, since, if $(a_1, a_2, \ldots) \cdot (b_1, b_2, \ldots) \in I_k$, then $a_i = 0$ and $b_j = 0$ for all $i \neq k$, $j \neq k$. Therefore, R is an example of a non-Noetherian ring with infinitely many minimal primes, and $(0) \subset R$ is an example of an ideal with infinitely many primes minimal over it.

3

Determine the Krull dimensions of the following rings: $\mathbb{R}[x, x^{-1}]$, $\mathbb{C}[x, y, z]/(z^2 - xy)$, $\mathbb{Z}[x, y]/(y^2 - x^3)$, $\mathbb{R}[x, y]/(x^2 + y^2 + 1)$, $\mathbb{Q}[x, y, z]/(y^2, z^3)$, $\mathbb{Q}[[x, y, z]]$, $\mathbb{Z}_{(2)}[x]$. Justify your assertions.

Answer: $\mathbb{R}[x, x^{-1}]$: Recall that the dimension of $\mathbb{R}[x]$ is 1, so, since any ideals in $\mathbb{R}[x, x^{-1}]$ must be ideals in $\mathbb{R}[x]$, we see that $\mathbb{R}[x, x^{-1}]$ has dimension at most 1. Furthermore, $\mathbb{R}[x, x^{-1}]$ is an integral domain, so (0) is prime in this ring. In addition, since $x - a \notin (x)$ for $a \neq 0$, $x - a$ is not a unit in $\mathbb{R}[x, x^{-1}]$, so $(x - a)$ is still an ideal in $\mathbb{R}[x, x^{-1}]$. $(x - a)$ is still prime in $\mathbb{R}[x, x^{-1}]$, and so we see that $(0) \subsetneq (x - a)$ is a strictly increasing chain of prime ideals, meaning that the dimension of $\mathbb{R}[x, x^{-1}]$ is at least 1. Since we showed it is at most 1 as well, we conclude that the dimension of $\mathbb{R}[x, x^{-1}]$ is 1.

$\mathbb{C}[x, y, z]/(z^2 - xy)$: Since $z^2 = xy$ defines a hypersurface in $\mathbb{C}^3$ and the dimension of $\mathbb{C}[x, y, z]$ is 3, we anticipate that $\mathbb{C}[x, y, z]/(z^2 - zy)$ should have dimension 2. Now, $(x - a)$ is prime in $\mathbb{C}[x, y, z]/(z^2 - xy)$ since it is prime in $\mathbb{C}[x, y, z]$ and the relation $z^2 = xy$ cannot reduce multiples of $x - a$. Furthermore, $(x - a, z - b)$ is prime in $\mathbb{C}[x, y, z]$ and again, the relation cannot reduce linear combinations of $x - a$ and $z - b$, so this ideal is minimal in $\mathbb{C}[x, y, z]/(z^2 - xy)$. However, the relations $x = a$, $z = b$ and $z^2 = xy$

entirely determine that

$$b^2 = ay \quad \Rightarrow \quad y = \frac{b^2}{a},$$

so $(\mathbb{C}[x,y,z]/(z^2 - xy))/(x-a, z-b) = \mathbb{C}[x,y,z]/(z^2 - xy, x-a, z-b) \simeq \mathbb{C}$, so $(x-a, z-b)$ is maximal. Similarly, $(x-a, y-b)$ entirely determines z and so is maximal. Since ideals of this form (i.e. $(x-a, y-b, z-c)$ with possibly one or more of these terms removed) comprise all the prime ideals in $\mathbb{C}[x,y,z]$, we see that we can form a chain of prime ideals no longer than 2 in $\mathbb{C}[x,y,z]/(z^2 - xy)$ (e.g. $(0) \subsetneq (x-a) \subsetneq (x-a, z-b)$), so the dimension of this ring is 2.

$\mathbb{Z}[x,y]/(y^2 - x^3)$: Since $\mathbb{Z}[x,y]$ is of dimension 3 and this situation is similar to the one above, we anticipate that the dimension of $\mathbb{Z}[x,y]/(y^2 - x^3)$ will also be 2. Since the relation $y^2 = x^3$ has no effect on constants, (3) is still a prime ideal in this ring. Also, since $\mathbb{Z}[x,y]/(y^2 - x^3)$ is an integral domain, (0) is still prime. Furthermore, $(3, x-2)$ is still prime in $\mathbb{Z}[x,y]/(y^2 - x^3)$, so we see that we have the chain of prime ideals $(0) \subsetneq (3) \subsetneq (3, x-2)$. Now, the relation $x = 2$ kills off both x and y, since $y^2 = x^3 = 2^3 = 8$. Thus,

$$(\mathbb{Z}[x,y]/(y^2 - x^3))/(3, x-2) \simeq \mathbb{Z}/3,$$

so $(3, x-2)$ is maximal. Thus, we conclude that $\mathbb{Z}[x,y]/(y^2 - x^3)$ is of dimension 3.

$\mathbb{R}[x,y]/(x^2 + y^2 + 1)$: Since $x^2 + y^2 + 1 = 0$ determines a hypersurface in $\mathbb{R}^2$ and $\mathbb{R}[x,y]$ has dimension 2, $\mathbb{R}[x,y]/(x^2 + y^2 + 1)$ should have dimension 1. Since this ring is still an integral domain, (0) is still prime. The minimal primes over (0) are just $(x-a)$ and $(y-b)$; arbitrarily, we pick (x). Then in

$$(\mathbb{R}[x,y]/(x^2 + y^2 + 1))/(x),$$

$y^2 = -1 - x^2 = -1 - 0 = -1$, so we see that this ring is just $\mathbb{R}[i]$, a field. Hence, (x) is maximal in $\mathbb{R}[x,y]/(x^2 + y^2 + 1)$, so $(0) \subsetneq (x)$ is a maximal chain, and so $\mathbb{R}[x,y]/(x^2 + y^2 + 1)$ has dimension 1.

$\mathbb{Q}[x,y,z]/(y^2, z^3)$: Since $z^3 = 0$ determines a hypersurface in $\mathbb{Q}^3$ and $y^2 = 0$ determines a hypersurface in $\mathbb{Q}^2$, we anticipate that this ring should have dimension 1. Now, $y \notin (0)$, but $y^2 = 0$ in this ring, so we see that (0) is not prime. Furthermore, since every ideal contains 0, we see that every prime ideal of $\mathbb{Q}[x,y,z]/(y^2, z^3)$ must contain y and z, so (y,z) is the minimal ideal in $\mathbb{Q}[x,y,z]/(y^2, z^3)$. Since

$$(\mathbb{Q}[x,y,z]/(y^2, z^3)/(y,z) = \mathbb{Q}[x,y,z]/(y,z) \simeq \mathbb{Q}[x]$$

which has dimension 1, we see that we can only build a chain of length 1 of prime ideals containing (y,z). One such would be $(y,z) \subsetneq (x,y,z)$, and so $\mathbb{Q}[x,y,z]/(y^2, z^3)$ has dimension 1.

$\mathbb{Q}[[x,y,z]]$: Since $\mathbb{Q}[[x]]$ has dimension 1, we anticipate that this ring should have dimension 3. Now, $\mathbb{Q}[[x,y,z]]$ is a local ring with unique maximal ideal (x,y,z), so any maximal chain of prime ideals must end with

(x, y, z). Since the only prime ideals contained in (x, y, z) are (x), (y) and (x, y), we see that the following is a maximal chain:

$$(0) \subsetneq (x) \subsetneq (x, y) \subsetneq (x, y, z)$$

((0) is prime since $\mathbb{Q}[[x, y, z]]$ is an integral domain). Therefore, $\mathbb{Q}[[x, y, z]]$ has dimension 3.

$\mathbb{Z}_{(2)}[x]$: In $\mathbb{Z}[x]$, the prime ideals are of the form (p), $(f(x))$ and $(p, f(x))$ for prime p and irreducible f. Now, since (2) is the only prime ideal in $\mathbb{Z}_{(2)}$, we see that in this ring we are restricted to $(2), (f(x))$ and $(2, f(x))$. Either way we slice it, we get a chain of the form $(0) \subsetneq (f) \subsetneq (2, f)$ or $(0) \subsetneq (2) \subsetneq (2, f)$, so we see that $\mathbb{Z}_{(2)}[x]$ has dimension 2.

♣

4

Given a commutative ring R, define the maximal spectrum of R (denoted Max R) to be the set of maximal ideals of R. For each subset $E \subset R$, let $V(E)$ denote the set $\{\mathfrak{m} \in \text{Max } R | E \subset \mathfrak{m}\} \subset \text{Max } R$.

(a): Show that Max R has a topology in which the closed sets are precisely the sets $V(E)$.

Proof. If we can show that sets of the form $V(E)$ are closed under arbitrary intersections and finite unions, then this will imply that their complements are closed under arbitrary unions and finite intersections, and so form a subbasis for a topology on Max R in which the closed sets are precisely the sets $V(E)$. This is exactly the conclusion arrived at by (d) and (e) below (given the result proved in (b)). □

(b): Show that $V(E) = V(I)$ for any $E \subset R$, where I is the ideal generated by E.

Proof. Suppose $\mathfrak{m} \in V(E)$. Then $\mathfrak{m} \supset I$, since the ideal generated by E is certainly contained in the ideal generated by all the elements of $\mathfrak{m}$, namely $\mathfrak{m}$. Hence, $V(E) \subset V(I)$. On the other hand, if $\mathfrak{m} \in V(I)$, then $\mathfrak{m} \supset I \supset E$, so $\mathfrak{m} \in V(E)$. Therefore $V(I) \subset V(E)$ and so, having shown containment in both directions, we conclude that $V(E) = V(I)$. □

(c): Show that $V(I) = V(\sqrt{I})$ for any ideal I.

Proof. If $\mathfrak{m} \in V(\sqrt{I})$, then $\mathfrak{m} \supset \sqrt{I} \supset I$, so $\mathfrak{m} \in V(I)$. Hence, $V(\sqrt{I}) \subset V(I)$. On the other hand, suppose $\mathfrak{m} \in V(I)$. Let $r \in \sqrt{I}$. Then $r^n \in I \subset \mathfrak{m}$ for some $n \in \mathbb{N}$. Since $\mathfrak{m}$ is maximal and, therefore, prime, this implies that $r \in \mathfrak{m}$. Thus, $\mathfrak{m} \supset \sqrt{I}$, and so we see that $V(I) \subset V(\sqrt{I})$; having shown containment both ways, we conclude that $V(I) = V(\sqrt{I})$. □

(d): Show that $V(\bigcup_\alpha E_\alpha) = \bigcap_\alpha V(E_\alpha)$ for any collection of subsets $\{E_\alpha\}_{\alpha\in A}$ and that $V(I_1 + \ldots + I_n) = V(I_1) \cap \cdots \cap V(I_n)$ for any ideals $I_1, \ldots, I_n$.

Proof. First, suppose $\mathfrak{m} \in V(\bigcup_\alpha E_\alpha)$. Then $\mathfrak{m} \supset \bigcup_\alpha E_\alpha$; in particular, $\mathfrak{m} \supset E_\beta$ for all β. Hence, $\mathfrak{m} \in V(E_\beta)$ for all β, and so $\mathfrak{m} \in \bigcap_\alpha V(E_\alpha)$. Hence, $V(\bigcup_\alpha E_\alpha) \subset \bigcap_\alpha V(E_\alpha)$.

On the other hand, if $\mathfrak{m} \in \bigcap_\alpha V(E_\alpha)$, then $\mathfrak{m} \in V(E_\beta)$ for all β. Hence, $\mathfrak{m} \supset E_\beta$ for all β, meaning that $\mathfrak{m} \supset \bigcup_\alpha E_\alpha$. Since our choice of $\mathfrak{m}$ was arbitrary, we see that $\bigcap_\alpha V(E_\alpha) \subset V(\bigcup_\alpha E_\alpha)$; having shown containment both directions, we conclude that $V(\bigcup_\alpha E_\alpha) = \bigcap_\alpha V(E_\alpha)$.

Now, suppose $\mathfrak{m} \in V(I_1+\ldots+I_n)$. Then $\mathfrak{m} \supset I_1+\ldots+I_n \supset I_i$ for all $i = 1, \ldots, n$, so $\mathfrak{m} \in V(I_i)$ for all i. Hence, $\mathfrak{m} \in V(I_1)\cap\cdots\cap V(I_n)$. Since our choice of $\mathfrak{m}$ was arbitrary, we see that $V(I_1 + \ldots + I_n) \subset V(I_1) \cap \cdots \cap V(I_n)$.

On the other hand, suppose $\mathfrak{m} \in V(I_1) \cap \cdots \cap V(I_n)$. Then $\mathfrak{m} \in V(I_i)$ for all $i = 1, \ldots, n$, so $\mathfrak{m} \supset I_i$ for all $i = 1, \ldots, n$. Hence, a generic element $\sum_{i=1}^n a_i$ in $I_1 + \ldots + I_n$ must be contained in $\mathfrak{m}$ since each of its summands are, so $\mathfrak{m} \in V(I_1 + \ldots + I_n)$ Since our choice of $\mathfrak{m}$ was arbitrary, we see that $V(I_1) \cap \cdots \cap V(I_n) \subset V(I_1 + \ldots + I_n)$; having shown containment both ways, we conclude that $V(I_1 + \ldots + I_n) = V(I_1) \cap \cdots \cap V(I_n)$. □

(e): Show that $V(I_1 \cap \cdots \cap I_n) = V(I_1) \cup \cdots \cup V(I_n)$ for any ideals $I_1, \ldots, I_n$ of R. Also explain the relationship with problem 1 above.

Proof. Suppose $\mathfrak{m} \in V(I_1)\cup\cdots\cup V(I_n)$. Then $\mathfrak{m} \in V(I_j)$ for some j and, hence, $\mathfrak{m} \supset I_j \supset I_1 \cap\cdots\cap I_n$. Hence, since our choice of $\mathfrak{m}$ was arbitrary, we see that $V(I_1) \cup \cdots \cup V(I_n) \subset V(I_1 \cap \cdots \cap I_n)$.

On the other hand, suppose $\mathfrak{m} \in V(I_1\cap\cdots\cap I_n)$. Then, since $\mathfrak{m}$ is maximal, it is also prime and so, by the result proved in PS8#8(a) from last semester, $\mathfrak{m} \supset I_j$ for some j. Hence, $\mathfrak{m} \in V(I_j) \subset V(I_1)\cup\cdots\cup V(I_n)$. Since our choice of $\mathfrak{m}$ was arbitrary, we see that $V(I_1\cap\cdots\cap I_n) \subset V(I_1)\cup\cdots V(I_n)$; having shown containment both directions, we conclude that $V(I_1 \cap \cdots \cap I_n) = V(I_1) \cup \cdots \cup V(I_n)$.

In problem 1 above, we proved essentially the same results as in (d) and (e) of this problem, except for points in a variety rather than for maximal ideals. However, by the Nullstellensatz, points in an affine variety and maximal ideals in the ring of functions are equivalent, so these are really the same result. □

(f): Give examples to illustrate (b)-(e) geometrically, in the case $R = \mathbb{R}[x]$, and in the case $R = \mathbb{Z}$.

Examples: Consider the subsets $A_1 = \{x - 1, x^2 - 1\}$ and $A_2 = \{x^2\}$ of $\mathbb{R}[x]$. Then $V(A_1) = \{(x - 1), (x + 1)\} = V(x - 1, x + 1)$ where $(x - 1, x^2 - 1)$ is the ideal generated by A_1. Similarly,

$V(A_2) = \{(x)\} = V(x^2)$ where (x^2) is the ideal generated by A_2. Note that $\sqrt{(x^2)} = (x)$ and $V(x) = \{(x)\} = V(x^2)$. Similarly, $\{(x-1),(x+1)\} = \sqrt{(x-1, x^2-1)}$. Now, $V(A_1 \cup A_2) = \emptyset = \{(x-1),(x+1)\} \cap \{(x)\}$. On the other hand, $V(\{x+1\} \cup \{x^2-1\}) = \{(x+1)\} = \{(x+1)\} \cap \{(x+1),(x-1)\}$, giving us an example for (d). In the case of (e), $V((x^2) \cap (x-1)) = V((x^3 - x^2)) = \{(x),(x-1)\} = \{(x)\} \cup \{(x-1)\} = V(x^2) \cup V(x-1)$. Note that in all of this we could simply have looked at the points at which each of these ideals vanish and come to the same conclusions, without knowing much of anything about ideals.

Turning to $\mathbb{Z}$, let $E_1 = \{2,4,6,8\}$ and $E_2 = \{6,12\}$. Then $V(E_1) = (2) = V(2)$ and (2) is the ideal generated by E_1. Similarly, $V(E_2) = \{(2),(3)\} = V(6)$, which is the ideal generated by E_2. Note that $(2) = \sqrt{(2)}$ and $(2) \cup (3) = \sqrt{(6)}$ and $V((2) \cup (3)) = \{(2),(3)\}$. Now, $V(E_1 \cup E_2) = V(\{2,4,6,8,12\}) = \{(2)\} = \{(2)\} \cap \{(2),(3)\}$. Finally, $V((2) \cap (6)) = V(6) = \{(2),(3)\} = \{(2)\} \cup \{(2),(3)\} = V(2) \cup V(6)$.

♣

(g): If $R = \mathbb{C}[x,y]/(f)$, is there a continuous bijective map between Max R and the locus of zeroes of f in $\mathbb{C}^2$? In which direction?

Answer: The Nullstellensatz tells us that there is indeed a bijection between Max R and the zeroes of f in $\mathbb{C}^2$. Specifically, we have the map $f : V(f) \to \text{Max } R$ given by

$$p \mapsto \mathfrak{m}_p,$$

where $\mathfrak{m}_p$ is the maximal ideal whose elements vanish at p. Now, we know that the closed sets of Max R are simply of the form $V(E)$; let $V(E)$ be such a closed set. Then, by part (b) above, $V(E) = V(I)$ where I is the ideal generated by E. Hence, $f^{-1}(V(E)) = f^{-1}(V(I))$, which consists precisely of the points at which all elements of I vanish. That is,

$$f^{-1}(V(E)) = f^{-1}(V(I)) = \bigcap_{g \in I} \{g^{-1}(0)\}.$$

Now, since 0 is closed in $\mathbb{C}$, $g^{-1}(0)$ is closed in $\mathbb{C}^2$ and, hence, $f^{-1}(V(I))$ is closed. Since our choice of closed set was arbitrary, we see that f is indeed a continuous map.

On the other hand, if we consider $\mathbb{C}[x,y]/(y)$, then $C = \{(z,0) \in \mathbb{C}^2 | |z| \leq 1\}$ is closed in $\mathbb{C}^2$. Now, we want to show that in this situation f^{-1} as defined above is not continuous; to do so, we demonstrate that $f(C)$ is not closed in Max R. Now,

$$f(C) = \{\mathfrak{m} = (x-a, y) | |a| \leq 1\} \neq \text{Max } R.$$

If $V(E) \supset f(C)$ and $f \in E$, then f must vanish on $(x-a, y)$ for all $|a| \leq 1$. This implies that for each such a, $c_{1a}(x-a) + c_{2a}y$ divides

f for some $c_{1a}, c_{2a} \in \mathbb{C}[x,y]$, so

$$f = \prod_{|a|\leq 1} (c_{1a}(x-a) + c_{2a}y)$$

Since this is impossible unless $c_{1a} = c_{2a} = 0$ (else f would be a polynomial of infinite degree), we see that $f \equiv 0$, and so $E = 0$. Thus, $V(E) = \text{Max } R$. Thus, the only closed set containing $f(C)$ is Max R, so $\overline{f(C)} = \text{Max } R$; however, since $f(C) \neq \text{Max } R$, this implies that $f(C)$ is not closed and so f^{-1} is not continuous. Therefore, we see that the map from Max R to the locus of zeroes of f, although bijective, is not, in general, continuous.

♣

5

Consider the following rings: $\mathbb{C}[x]$, $\mathbb{C}[x,y]$, $\mathbb{C}[x,y]/(x^2+y^2-1)$, $\mathbb{C}[x,y]/(x^2-y^2)$, $\mathbb{C}[x]/(x^2)$, $\mathbb{C}[x,y]/(x^2)$, $\mathbb{C}$, $\mathbb{C}\times\mathbb{C}$, $\mathbb{C}[x]/(x^2-x)$, $\mathbb{Z}/2$, $\mathbb{Z}/6$, $\mathbb{Z}$, $\mathbb{Z}[1/15]$. For each of them, do the following:

- **(a):** Describe all the maximal ideals in the given ring R, and describe Max R geometrically.
- **(b):** Determine whether Max R is connected.

Answer: $\mathbb{C}[x]$: The maximal ideals in $\mathbb{C}[x]$ are simply given by (f) where f is irreducible. Since $\mathbb{C}$ is algebraically closed, such an f must be of the form $x-a$, so the $(x-a)$ are the maximal ideals. Now, by the 1-dimensional analogue of 4(g) above, we know that the maximal ideals are in bijection with the zero locus of (0), namely all of $\mathbb{C}$, so geometrically Max $\mathbb{C}[x]$ just looks like $\mathbb{C}$. Furthermore, again by 4(g), we have a continuous map from $\mathbb{C}$ to Max $\mathbb{C}[x]$; since $\mathbb{C}$ is connected, Max $\mathbb{C}[x]$ must be connected as well.

$\mathbb{C}[x,y]$: The maximal ideals of $\mathbb{C}[x,y]$ are simply (f,g) where f and g are irreducible and relatively prime. Since we have, by 4(g), a bijection between the zero locus of (0) (i.e. $\mathbb{C}^2$) and Max $\mathbb{C}[x,y]/(0) = \mathbb{C}[x,y]$, we see that, geometrically, Max $\mathbb{C}[x,y]$ just looks like $\mathbb{C}^2$. Since $\mathbb{C}^2$ is connected and we have a continuous map from $\mathbb{C}^2$ to Max $\mathbb{C}[x,y]$, we see that Max $\mathbb{C}[x,y]$ is connected.

$\mathbb{C}[x,y]/(x^2+y^2-1)$: The zero locus of x^2+y^2-1 is just the sphere, so we know, by 4(g), that there is a bijection between the sphere and Max R. Hence, the maximal ideals of R are of the form $(x-a, y-b)$ where $|a^2+b^2| = 1$. This bijection tells us that Max R is geometrically just the sphere $\{(z,w) \in \mathbb{C}^2 | |z^2+w^2| = 1\}$; since this space is connected, Max R must be connected.

$\mathbb{C}[x,y]/(x^2-y^2)$: The zero locus of x^2-y^2 is simply given by the two complex lines $y = x$ and $y = -x$; points on these lines are simply of the form $(-a,a)$ and (a,a), respectively and so the maximal ideals of this ring are $(x+a, y-a)$ and $(x-a, y-a)$. Geometrically, then, Max R just looks like the union of the lines $y = x$ and $y = -x$. Since each of these lines is connected and $(0, 0)$ lies on both, this union is connected and so Max R must be connected.

$\mathbb{C}[x]/(x^2)$: Since this quotient imposes the relation $x^2 = 0$ on $\mathbb{C}[x]$, we see that, for $a \neq 0$, $(x - a)(x + a) = x^2 - a^2 = a^2$, which is a unit, so, since the maximal ideals in $\mathbb{C}[x]$ are of the form $(x - a)$, we see that the only maximal ideal of $\mathbb{C}[x]/(x^2)$ is (x). Hence, Max R corresponds geometrically to the single point $x = 0$, which is certainly connected.

$\mathbb{C}[x, y]/(x^2)$: The maximal ideals of this ring are in bijection with the zero locus of x^2; that is, the set $\{(0, a) \in \mathbb{C}^2\}$. Hence, the maximal ideals are simply $(x, y - a)$ for $a \in \mathbb{C}$. Geometrically, then, Max R just looks like the complex y-axis, which is connected, so Max R is connected.

$\mathbb{C}$: Since $\mathbb{C}$ is a field, the only ideal (and, hence, the only maximal ideal) is (0). Hence, Max $\mathbb{C}$ is just the single point (0), which is certainly connected.

$\mathbb{C}\times\mathbb{C}$: The only ideals of $\mathbb{C}\times\mathbb{C}$ are $(0, a) = (0, 1)$ and $(a, 0) = (1, 0)$. Since $(0, 1)$ does not contain $(1, 0)$ and *vice versa*, we see that $(1, 0) = V(1, 0)$ and $(0, 1) = V(0, 1)$ and so $(1, 0)$ and $(0, 1)$ are each both open and closed in Max R, so Max R is a set consisting of 2 points under the discrete topology, which is not connected.

$\mathbb{C}[x]/(x^2 - x)$: The zero locus of $x^2 - x$ is given by the two points $x = 0$ and $x - 1 = 0$, so, by the 1-dimensional analogue of 4(g), the maximal ideals of R are simply (x) and $(x - 1)$. Since neither of these two ideals contains the other, we see that $(x) = V(x)$ and $(x-1) = V(x-1)$, so Max R consists of two points under the discrete topology, which is not connected.

$\mathbb{Z}/2$: Since $\mathbb{Z}/2$ is a field, the only ideal (and, hence, the only maximal ideal) is (0). Hence, Max $\mathbb{Z}/2$ consists of a single point, which is certainly connected.

$\mathbb{Z}/6$: The only ideals of $\mathbb{Z}/6$ are $(0), (2)$ and (3). Since (3) does not contain (2) and *vice versa*, (2) and (3) are both maximal and, furthermore, $(2) = V(2)$ and $(3) = V(3)$, so Max $\mathbb{Z}/6$ is a two-point set with the discrete topology, which is not connected.

$\mathbb{Z}$: The maximal ideals of $\mathbb{Z}$ are of the form (p) for p prime. Now, if $E \subset \mathbb{Z}$, then $V(E) = V(I)$, where I is the ideal generated by E. Since $\mathbb{Z}$ is a PID, $I = (a)$ for some $a \in \mathbb{Z}$. Let $a = p_1^{i_1} \cdots p_n^{i_n}$ be the prime factorization of a; then $(a) \subset (p_i)$ for $i = 1, \ldots, n$ and, since this factorization is unique up to multiplication by a unit, we see that

$$V(E) = V(I) = \{(p_i)|i = 1, \ldots, n\}.$$

Since sets of this form are the only closed sets in Max $\mathbb{Z}$ and, furthermore, any finite set of Max $\mathbb{Z}$

$$\{(q_j)|j = 1, \ldots, m\} = V(q_i \cdots q_m)$$

for q_j prime, we see that Max $\mathbb{Z}$ is a topological space consisting of points corresponding to the prime numbers in $\mathbb{Z}$ with the finite complement topology. If Max $\mathbb{Z}$ were disconnected, then there would exist two non-empty, disjoint open sets A and B such that $A \cup B =$ Max $\mathbb{Z}$. However, for this to be the case, both A and B must be both open and closed and at least one must be infinite; since only finite subsets of Max $\mathbb{Z}$ are closed, this is impossible, so we see that Max $\mathbb{Z}$ is connected.

$\mathbb{Z}[1/15]$: The maximal ideals of $\mathbb{Z}[1/15]$ consist of those ideals of the form (p) where p is prime in $\mathbb{Z}$ and $p \neq 3$, $p \neq 5$. This is the case because $\mathbb{Z}[1/15]$ is simply an extension of $\mathbb{Z}$ in which 15 (and, hence, all divisors and multiples of 15) has been inverted. So doing does not introduce any new maximal ideals, it simply kills off those maximal ideals generated by the divisors of 15, namely (3) and (5). Furthermore, if $\frac{r}{s}$ is an element of $\mathbb{Z}[1/15]$ expressed in lowest terms, then we can still factor r into a product of primes (that aren't 3 or 5), and so the argument given above for $\mathbb{Z}$ carries over to demonstrate that Max $\mathbb{Z}[1/15]$ consists of points corresponding to the non-three, non-five primes of $\mathbb{Z}$ under the finite complement topology. Again, this means that Max $\mathbb{Z}[1/15]$ is connected.

♣

6

(a): Let $R = \mathbb{C}[x,y]/(x^2 - y^2)$ and $S = \mathbb{C}[x,y]/(x^2 - x)$. Show that there is a homomorphism $f : R \to S$ given by $f(x) = y - 2xy$, $f(y) = y$. Show that there is an induced continuous map $f^* :$ Max $S \to$ Max R given by $\mathfrak{m} \mapsto f^{-1}(\mathfrak{m})$. Describe the map f^* geometrically. Is it injective? Surjective?

Proof. Note, first of all, that $f(0) = 0$ (and, in fact, $f(c) = c$ for all $c \in \mathbb{C}$). Additionally,

$$f(x^2) = (y-2xy)^2 = y^2 - 4xy^2 + 4x^2y^2 = y^2 - 4x^2y^2 + 4x^2y^2 = y^2 = f(y^2),$$

so f preserves the relation $x^2 - y^2 = 0$. Since we're simply defining f on the generators of R and extending linearly, this means that f is indeed a homomorphism.

Now, the maximal ideals in $\mathbb{C}[x,y]/(x^2-y^2)$, as we saw in problem 5 above, are simply those ideals of the form $(x-a, y-a)$ and $(x+a, y-a)$. In $\mathbb{C}[x,y]/(x^2-x)$, the maximal ideals are simply of the form $(x, y-a)$ or $(x-1, y-a)$ (since (x) and $(x-1)$ are the maximal ideals in $\mathbb{C}[x]/(x^2-x)$ as we saw in problem 5). Now, for any $a \in \mathbb{C}$, $f^*(x, y-a) = f^{-1}(x, y-a)$ is an ideal in $\mathbb{C}[x,y]/(x^2-y^2)$ since f is a homomorphism. Furthermore,

$$f(x-a) = y - 2xy - a = (y-a) - 2yx \in (x, y-a)$$

and

$$f(y-a) = y - a \in (x, y-a),$$

so we see that $(x-a, y-a) \subset f^*(x, y-a)$. Since $(x-a, y-a)$ is maximal and $1 \notin f^*(x, y-a)$ (since $f(1) = 1 \notin (x, y-a)$), this implies that $f^*(x, y-a) = (x-a, y-a)$.

On the other hand, $f^*(x-1, y-a)$ is also an ideal; we want to show that $f^*(x-1, y-a) = (x+a, y-a)$. To that end, note that

$$f(x+a) = y - 2xy + a,$$

which vanishes at $x = 1, y = a$, since $a - 2(1)(a) + a = a - 2a + a = 0$, so $f(x+a) \in (x-1, y-a)$. On the other hand,

$$f(y-a) = y - a \in (x-1, y-a),$$

so we see that $(x+a, y-a) \subset f^*(x-1, y-a)$. Since $(x+a, y-a)$ is maximal and $1 \notin f^*(x-1, y-a)$, this means that $f^*(x-1, y-a) = (x+a, y-a)$.

Therefore, since we've examined all possible cases, we see that $f^* : \text{Max } S \to \text{Max } R$ is well-defined. Furthermore, suppose $C \subset \text{Max } R$ is closed. Then there exists $E \subset R$ such that $C = V(E)$. Thus,

$$f^{*-1}(C) = f^{*-1}(V(E)) = \{\mathfrak{m} \in \text{Max } S | f^*(\mathfrak{m}) \supset E\}.$$

Now, let $\mathfrak{m} \in V(f(E))$; i.e. $\mathfrak{m} \supset f(E)$. Then

$$f^*(\mathfrak{m}) = f^{-1}(\mathfrak{m}) \supset f^{-1}(f(E)) \supset E,$$

so we see that $\mathfrak{m} \in f^{*-1}(V(E))$. Since our choice of $\mathfrak{m}$ was arbitrary, we see that $V(f(E)) \subset f^{*-1}(V(E))$.

On the other hand, suppose $\mathfrak{m} \in f^{*-1}(V(E))$. Then $f^{-1}(\mathfrak{m}) = f^*(\mathfrak{m}) \supset E$, which implies that

$$\mathfrak{m} = f(f^{-1}(\mathfrak{m})) \supset f(E),$$

and so $\mathfrak{m} \in V(f(E))$. Since our choice of $\mathfrak{m}$ was arbitrary, we see that $f^{*-1}(V(E)) \subset V(f(E))$; having shown containment both ways, we see that $f^{*-1}(V(E)) = V(f(E))$. Specifically, this means that $f^{*-1}(V(E))$ is closed. Since our choice of $C = V(E)$ was arbitrary, we see that the preimage of all closed sets under f^* is closed, and so f^* is continuous.

To describe this map geometrically, recall that we have a bijection between Max R (respectively Max S) and the zero locus of $x^2 - y^2$ (resp. $x^2 - x$); drawing those in the xy-plane yields:

where f^* corresponds to the arrows pictured. Now, from this picture it is clear that f^* is not injective; specifically, $f^*(x-1, y) = f^*(x, y) = (x, y)$. On the other hand, in the process of demonstrating that f^* was well-defined, we actually showed that f^* was surjective, since $f^*(x, y-a) = (x-a, y-a)$ and $f^*(x-1, y-a) = (x+a, y-a)$, which comprise all maximal ideals of R; the picture bears this out. □

(b): In general, if $f : R \to S$ is a homomorphism of commutative rings, is there an induced continuous map $f^* : \text{Max } S \to \text{Max } R$? What if we instead considered the *prime spectrum* of R and of S?

Answer: Consider the rings $R = \mathbb{Z}$ and $S = \mathbb{Q}$. Let $i : \mathbb{Z} \to \mathbb{Q}$ be the inclusion. Then i is certainly a ring homomorphism. Now, since $\mathbb{Q}$ is a field, the only proper ideal of $\mathbb{Q}$ is (0), so $\text{Max } \mathbb{Q} = \{(0)\}$. Now, $i^{-1}(0) = (0)$ since i is an injection. However, (0) is not maximal in $\mathbb{Z}$, so we see that the desired map $i^* : \text{Max } \mathbb{Q} \to \text{Max } \mathbb{Z}$ is not even well-defined.

On the other hand, consider rings R and S and homomorphism $f : R \to S$. Let $\mathfrak{p} \in \text{Spec } S$. Then, since f is a homomorphism, $f^{-1}(\mathfrak{p})$ is an ideal. Now, suppose $a, b \in R$ such that $ab \in f^{-1}(\mathfrak{p})$. Then

$$f(a)f(b) = f(ab) \in \mathfrak{p},$$

meaning that $f(a) \in \mathfrak{p}$ or $f(b) \in \mathfrak{p}$. Hence, either $a \in f^{-1}(\mathfrak{p})$ or $b \in f^{-1}(\mathfrak{p})$, so we see that $f^{-1}(\mathfrak{p})$ is prime. Hence, $f^* : \text{Spec } S \to \text{Spec } R$ given by

$$\mathfrak{p} \mapsto f^{-1}(\mathfrak{p})$$

is well-defined.

Now, suppose $C \subset \text{Spec } R$ is closed. Then there exists $E \subset R$ such that $C = V(E)$. Thus,

$$f^{*-1}(C) = f^{*-1}(V(E)) = \{\mathfrak{p} \in \text{Spec } S | f^*(\mathfrak{p}) \supset E\}.$$

Now, let $\mathfrak{p} \in V(f(E))$; i.e. $\mathfrak{p} \supset f(E)$. Then

$$f^*(\mathfrak{p}) = f^{-1}(\mathfrak{p}) \supset f^{-1}(f(E)) \supset E,$$

so we see that $\mathfrak{p} \in f^{*-1}(V(E))$. Since our choice of $\mathfrak{p}$ was arbitrary, we see that $V(f(E)) \subset f^{*-1}(V(E))$.

On the other hand, suppose $\mathfrak{p} \in f^{*-1}(V(E))$. Then $f^{-1}(\mathfrak{p}) = f^*(\mathfrak{p}) \supset E$, which implies that

$$\mathfrak{p} = f(f^{-1}(\mathfrak{p})) \supset f(E),$$

and so $\mathfrak{p} \in V(f(E))$. Since our choice of $\mathfrak{p}$ was arbitrary, we see that $f^{*-1}(V(E)) \subset V(f(E))$; having shown containment both ways, we see that $f^{*-1}(V(E)) = V(f(E))$. Specifically, this means that $f^{*-1}(V(E))$ is closed. Since our choice of $C = V(E)$ was arbitrary, we see that the preimage of all closed sets under f^* is closed, and so f^* is continuous.

Now, $f : R \to S$ induces a natural R-algebra structure on S. By criterion (iii) from PS8#8, this means that f^* is surjective if and only if S is a faithfully flat R-module. Since not all R-algebras are faithfully flat, we see that f^* is not necessarily surjective. Furthermore, part (a) above gives a counterexample to the notion that f^* must always be injective, since the maximal ideals $(x-1, y)$ and (x, y) are also necessarily prime and are both mapped to the same element by f^* (which was defined in (a) as a map from Max S to Max R, but, in being so defined, is simply the restriction of $f^* : \text{Spec } S \to \text{Spec } R$).

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