ALGEBRA HW 6

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1

(a): Let R be a Noetherian ring, $\mathcal{I}$ the set of ideals of R, and $\mathcal{I}_0$ a subset of $\mathcal{I}$. Let P be a property that ideals in $\mathcal{I}_0$ may or may not have. Suppose that one can show the following condition:

(1) $\forall I \in \mathcal{I}_0$, if every ideal $J \in \mathcal{I}_0$ that properly contains I has property P, then so does I.

Conclude that P holds for *all* $I \in \mathcal{I}_0$.

Proof. Suppose, for the sake of deriving a contradiction, that P does not hold for some I in $\mathcal{I}_0$. Then, since we're assuming we can show condition (1), this implies that some $I' \in \mathcal{I}_0$ properly containing I also doesn't satisfy property P. In turn, since P does not hold for I', this implies that P does not hold for some $I'' \in \mathcal{I}_0$ properly containing I'. Iterating this process yields:

$$I \subsetneq I' \subsetneq I'' \subsetneq I''' \subsetneq \cdots,$$

an infinite ascending chain in $\mathcal{I}_0$ and, hence, in R. However, since R is Noetherian, this is impossible. Thus, we see that, in fact, P must hold for every $I \in \mathcal{I}_0$. □

(b): Use this principle ("Noetherian induction") to prove that if R is a Noetherian integral domain, and $r \in R$ is a non-zero non-unit, then r is a product of irreducible elements of R.

Proof. Let $r \in R$ be a non-zero non-unit, and let $\mathcal{I}_0$ be the set of all principal ideals in R containing (r). For $(a) \in \mathcal{I}_0$, let $P(a)$ be the property that a is the product of irreducibles. Let $(b) \in \mathcal{I}_0$ and suppose $P(c)$ is true for all $(c) \in \mathcal{I}_0$ such that $(b) \subsetneq (c)$. Then either b is irreducible or not. If b is irreducible, then clearly $P(b)$ is true. If b is not irreducible, then $b = cd$ for some $c, d \in R$ where c and d are non-units. Then $c, d \notin (b)$ but $b \in (c)$ and $b \in (d)$, so $(r) \subset (b) \subsetneq (c)$ and $(r) \subset (b) \subsetneq (d)$, so (c) and (d) are elements of $\mathcal{I}_0$ properly containing (b) and thus, by hypothesis, $P(c)$ and $P(d)$ are true. Hence, $c = r_1 r_2 \cdots r_n$ and $d = s_1 s_2 \cdots s_m$ for irreducible r_i and s_j. Hence,

$$b = cd = r_1 \cdots r_n s_1 \cdots s_m,$$

so b is the product of irreducibles.

Since our choice of (b) was arbitrary, we see that for all $(a) \in \mathcal{I}_0$, if $P(c)$ is true for all $(c) \in \mathcal{I}_0$ properly containing (a), then $P(a)$ is true.

Therefore, by our result from part (a) above, we conclude that for all $(a) \in \mathcal{I}_0$, $P(a)$ is true; i.e. a is the product of irreducibles. Since (r) contains itself and is principal, $(r) \in \mathcal{I}_0$, so r is the product of irreducibles. In turn, since our choice of r was arbitrary, we see that any non-zero non-unit in R is the product of irreducibles. □

(c): Show that (b) (and therefore (a)) fails in general if R is not Noetherian.

2

Let $f : Y \to X$ be a polynomial map of complex affine varieties, corresponding to a ring extension $i : A \hookrightarrow B$. Suppose that B is an integral extension of A. Show that the map f is closed in the Zariski topology.

Proof. Let $V \subseteq Y$ be closed. Then $V = V(J)$ for some ideal $J \subseteq B$. Now, we want to show that $f(V)$ is closed in X; specifically, that $f(V) = V(i^{-1}(J))$. Now, since B is an integral extension of A, if $\mathfrak{n}$ is a maximal ideal containing J, then $\mathfrak{n} \cap A$ is maximal in A and contains $I^{-1}(J) = J \cap A$. Hence, $f(V) \subset V(i^{-1}(J))$.

On the other hand, if $\mathfrak{m}$ is a maximal ideal containing $i^{-1}(J)$, then, since $\mathfrak{m}$ is prime, $\mathfrak{m}$ is the contraction of $\mathfrak{q} \subset B$, where $\mathfrak{q}$ is prime. Then, for any maximal $\mathfrak{n} \supset \mathfrak{q}$, $\mathfrak{n} \cap A$ is maximal (since $B \supset A$ is integral) and contains $\mathfrak{m}$, so $\mathfrak{n} \cap A = \mathfrak{m}$. Now, let $N = \{\mathfrak{n} | \mathfrak{n} \cap A = \mathfrak{m}\}$.

Show some element of N contains J. Then this implies that every maximal ideal containing $i^{-1}(J)$ is the contraction of a maximal ideal in B containing J, so $V(i^{-1}(J)) \subset f(V)$. Therefore, since we've shown containment in both directions, $f(V) = V(i^{-1}(J))$ is closed; since our choice of closed V was arbitrary, we conclude that f is a closed map. □

3

(a): Let $A = \mathbb{C}[x]$, $B = \mathbb{C}[x,y]/(xy-1)$. Is B integral over A? Describe the corresponding map on varieties. Is it closed?

(b): Do the same with B replaced by $\mathbb{C}[x,y]/(x^2+y^2-1)$.

Answer: Note that

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(c): Do the same with B replaced by $\mathbb{C}[x,y]/(y^2-x)$.

(d): Do the same with B replaced by $\mathbb{C}[x, y, \frac{1}{y-1}]/(y^2-x)$.

Answer: We claim that B is not integral because $\frac{1}{y-1}$ is not integral over A. To see this, note that, since $y^2 = x$,

$$(y-1)^2 = y^2 - 2y + 1 = x - 2y + 1,$$

so

$$(y-1)^2 + 2(y-1) - (x-1) = x - 2y + 1 + 2y - 2 - x + 1 = 0,$$

so $y-1$ satisfies the polynomial $t^2 + 2t - (x-1)$. Hence, $\frac{1}{y-1}$ satisfies

$$t^2\left[\left(\frac{1}{t}\right)^2 + \frac{2}{t} - (x-1)\right] = 1 + 2t - (x-1)t^2,$$

which is not monic. If $\frac{1}{y-1}$ is integral over A, then there must be some monic factor of this polynomial which is satisfied by $\frac{1}{y-1}$; however $\frac{1}{y-1} \notin \mathbb{C}[x]$, so this is impossible. Hence, we conclude that $\mathbb{C}[x, y, \frac{1}{y-1}]/(y^2 - x)$ is not integral over $\mathbb{C}[x]$.

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4

Let n be a square-free non-zero integer. Let R_n be the integral closure of $\mathbb{Z}$ in $\mathbb{Q}(\sqrt{n})$. Show that $R_n = \mathbb{Z}\left[\frac{1+\sqrt{n}}{2}\right]$ if $n \equiv 1 \pmod 4$, and that $R_n = \mathbb{Z}[\sqrt{n}]$ otherwise.

Proof. Let $a + b\sqrt{n}$ be an element of $\mathbb{Q}(\sqrt{n})$ that is integral over $\mathbb{Z}$. Then $a+b\sqrt{n}$ is the root of a monic quadratic polynomial in $\mathbb{Z}[x]$. Since $x+y\sqrt{n} \mapsto x - y\sqrt{n}$ defines an automorphism of $\mathbb{Q}(\sqrt{n})$ fixing $\mathbb{Z}$, $a - b\sqrt{n}$ must be a root of any polynomial in $\mathbb{Z}[x]$ with $a + b\sqrt{n}$ as a root. Hence, the monic quadratic polynomial we're talking about is

$$(x - (a + b\sqrt{n}))(x - (a - b\sqrt{n})) = x^2 - 2ax + a^2 - b^2 n.$$

Therefore, $2a \in \mathbb{Z}$, so $a = \frac{p}{2}$ for some $p \in \mathbb{Z}$. Now, since $a^2 - b^2 n \in \mathbb{Z}$, we must have that

$$a^2 - b^2 n = \frac{p^2 - 4b^2 n}{4} \in \mathbb{Z},$$

or $p^2 - 4b^2 n \equiv 0 \pmod 4$, which is to say that $4b^2 n \equiv p^2 \pmod 4$. Now, $b = \frac{r}{s}$ for some $r, s \in \mathbb{Z}$. Since $4b^2 n$ is an integer, either $s = 1$ or s^2 divides n or 4; since n is square-free, this second case requires that $s^2 = 4$, so $s = \pm 2$. If $s = 1$, then $b \in \mathbb{Z}$. If $s = 2$, then $4b^2 = 4\frac{r^2}{4} = r^2$, so

$$4b^2 n = r^2 n \equiv p^2 \pmod 4.$$

If p is odd, then $p^2 \equiv 1 \pmod 4$; since the only squares mod 4 are 0 and 1, this implies that both r^2 and n are equivalent to 1 modulo 4. In turn, this means that r is odd. If p is even, then $p^2 \equiv 0 \pmod 4$; since n is square-free, $n \not\equiv 0 \pmod 4$, so it must be the case that $r^2 \equiv 0 \pmod 4$, which is to say that r is even.

Now recall that $a = \frac{p}{2}$ and $b = \frac{r}{2}$. We've just seen that if $n \equiv 1 \pmod 4$, then p and r can be any integers, so we see that the integral elements $a+b\sqrt{n}$ consist precisely of the elements of $\mathbb{Z}\left[\frac{1+\sqrt{n}}{2}\right]$, so this is the integral closure of $\mathbb{Z}$ in $\mathbb{Q}(\sqrt{n})$. On the other hand, we saw that if $n \not\equiv 1 \pmod 4$, then r and p are both even, so a and b are both integers; hence, the integral closure of $\mathbb{Z}$ in $\mathbb{Z}(\sqrt{n})$ is just $\mathbb{Z}[\sqrt{n}]$. □

5

For each of the following rings R, determine whether R has a height one prime that is not principal. If there is one, find one explicitly. If there isn't one, determine whether there is *some* prime ideal that is not principal, and find one explicitly if it exists.

(a): $\mathbb{Z}[i, x, y]$.

Answer: Recall that a ring is a UFD if and only if every height one prime is principal. Since $\mathbb{Z}[i]$ is a UFD, so is $\mathbb{Z}[i][x, y] = \mathbb{Z}[i, x, y]$, so every height one prime in $\mathbb{Z}[i, x, y]$ is principal. However, if we take

$$\mathbb{Z}[i, x, y]/(x, y) \simeq \mathbb{Z}[i]$$

which is an integral domain, so (x, y) is a prime ideal. Since (x, y) is not principal, we see that this is an example of a prime ideal that is not principal.

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(b): $\mathbb{Q}[x, y, z, w]/(xy - zw)$.

Answer: Consider the ideal (x, z). Now, if we mod out by (x, z), then we're essentially just saying that $x = z = 0$, so $xy = 0 = zw$, which kills off the relation induced by modding out by $(xy - zw)$. Hence,

$$(\mathbb{Q}[x, y, z, w]/(xy - zw))/(x, z) \simeq \mathbb{Q}[x, y, z, w]/(x, z) \simeq \mathbb{Q}[y, w],$$

which is an integral domain, so (x, z) is a prime ideal.

Since $\mathbb{Q}[x, y, z, w]/(xy - zw)$ is an integral domain, (x, z) is height one if and only if the only prime ideal contained in (x, z) is (0). Now, (x) is not prime in $\mathbb{Q}[x, y, z, w]/(xy - zw)$, since $xy = zw$ is in (x), but neither z nor w are in (x). Similarly, $zw = xy$ is in (z) but neither x nor y is in (z), so (z) is not prime either. Thus, we conclude that the only prime ideal contained in (x, z) is (0), so (x, z) is a height one prime that is not principal.

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(c): $\mathbb{Z}[\sqrt{-5}]$.

Answer: Consider the ideal $(2, 1 + \sqrt{-5})$. Now, $\mathbb{Z}[\sqrt{-5}]/(1 + \sqrt{-5}) \simeq \mathbb{Z}$, so

$$\mathbb{Z}[\sqrt{-5}]/(2, 1 + \sqrt{-5}) \simeq (\mathbb{Z}[\sqrt{-5}]/(1 + \sqrt{-5}))/(2) \simeq \mathbb{Z}/(2) = \mathbb{Z}/2,$$

which is a field and, thus, an integral domain, so $(2, 1 + \sqrt{-5})$ is prime.

Now, since $\mathbb{Z}[\sqrt{-5}]$ is an integral domain, $(2, 1+\sqrt{-5})$ is height one only if the only prime ideal it contains is (0). (2) is certainly not prime, since $6 = 2 \cdot 3 \in (2)$, but

$$6 = (1+\sqrt{-5})(1-\sqrt{-5})$$

also, and $1 \pm \sqrt{-5} \notin (2)$. Similarly, $6 \in (1+\sqrt{-5})$ but neither 2 nor 3 is, so $(1+\sqrt{-5})$ is not prime. We conclude that the only prime contained in $(2, 1+\sqrt{-5})$ is (0), so $(2, 1+\sqrt{-5})$ is a non-principal height one prime.

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(d): $\mathbb{Z}[x, y]/(5, y - x^3 - x + 1)$.

Answer: Note that

$$\mathbb{Z}[x, y]/(5, y - x^3 - x + 1) \simeq \mathbb{Z}/5[x][y]/(y - x^3 - x + 1).$$

Now, as a polynomial in $\mathbb{Z}/5[x][y]$, $y - x^3 - x + 1$ is a linear polynomial with non-zero constant term and, hence, irreducible. Therefore,

$$\mathbb{Z}/5[x][y]/(y - x^3 - x + 1) \simeq \mathbb{Z}/5[x],$$

which is a Euclidean domain and, therefore, a PID. Hence, every ideal (including primes) is principal in this ring, so there are no non-principal primes and certainly no non-principal height one primes.

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