ANALYSIS HW 1

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1

Newton's method for computing the square root of a positive real number a uses

$$x_{n+1} = \frac{1}{2}(x_n + \frac{a}{x_n}).$$

Show that with some appropriate initial guess x_0 this sequence converges to $\sqrt{a}$.

Proof. Let $x > \sqrt{a}$. Then

$$\begin{aligned} x^2 &> a && \text{divide both sides by } x \\ x &> \tfrac{a}{x} && \text{or} \\ x &> \tfrac{1}{2}(x + \tfrac{a}{x}) && \text{since } s + \tfrac{a}{s} < 2x \end{aligned}$$

Now, clearly,

$$\begin{aligned} (x - \tfrac{a}{x})^2 &> 0 && \text{which can be expanded to see} \\ x^2 - 2a + \tfrac{a^2}{x^2} &> 0 && \text{then, adding } 4a \text{ to both sides} \\ x^2 + 2a + \tfrac{a^2}{x^2} &> 4a && \text{or, taking a square root} \\ \tfrac{1}{2}(x + \tfrac{a}{x}) &> \sqrt{a} \end{aligned}$$

In other words, for $x > \sqrt{a}$,

$$\sqrt{a} < \frac{1}{2}(x + \frac{a}{x}) < x.$$

Hence, if we let $x_0 > \sqrt{a}$, the sequence $\{x_n\}$ is strictly decreasing and bounded below by $\sqrt{a}$. Thus, this sequence converges on $[\sqrt{a}, +\infty)$.

Now, to see that the sequence converges to $\sqrt{a}$, it suffices to show that $x_n^2 \to a$. Let $\epsilon > 0$. Since $\{x_n\}$ is Cauchy, there exists $N \in \mathbb{N}$ such that, for $k, l > N$,

$$||x_k - x_l|| < \sqrt{\epsilon}.$$

Let $j > N$. Then

$$\begin{aligned}
||x_{j+1}^2 - a|| &= ||\tfrac{1}{4}(x_j^2 + 2a + \tfrac{a^2}{x_j^2}) - a|| \\
&= ||\tfrac{1}{4}(x_j^2 - 2a + \tfrac{a^2}{x_j^2})|| \\
&= ||[\tfrac{1}{2}(-x_j + \tfrac{a}{x_n})]^2|| \\
&= ||[\tfrac{1}{2}(x_j + \tfrac{a}{x_n}) - x_j]^2|| \\
&= ||(x_{j+1} - x_j)^2|| \\
&= ||x_{j+1} - x_j||^2 \\
&< \epsilon.
\end{aligned}$$

Therefore the sequence $\{x_n\}$ converges to $\sqrt{a}$ so long as $x_0 > \sqrt{a}$. $\square$

2

In a metric space, if a subsequence of a Cauchy sequence x_j converges to some point x, show that the whole sequence also converges to x.

Proof. Let $\{x_j\}$ be a Cauchy sequence in the metric space M that contains a subsequence $\{x_{j_k}\}$ which converges to x. Let $\epsilon > 0$. Since $\{x_j\}$ is Cauchy, there exists $N_1 \in \mathbb{N}$ such that, for all $m, n \geq N_1$,

$$d(x_m, x_n) < \frac{\epsilon}{2}.$$

Also, since $\{x_{j_k}\}$ converges to x, there exists $N_2 \in \mathbb{N}$ such that, for all $r > N_2$,

$$d(x_{j_r}, x) < \frac{\epsilon}{2}.$$

Define $N_3 = j_{N_2} \geq N_2$ and let

$$N = \max\{N_1, N_3\}.$$

If $s > N$, then

$$d(x_s, x) \leq d(x_s, x_{j_t}) + d(x_{j_t}, x) \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

where $j_t > N$. $\square$

3

In a normed linear space V, an infinite series $\sum x_j$ is said to *converge absolutely* if the series of real numbers $\sum ||x_j||$ converges.

a) If V is complete, show that a series that converges absolutely also converges.

Proof. Let $\sum x_j$ be an absolutely convergent series in V. Then we know that the sequence of partial sums $\sum_{i=1}^{k} ||x_i||$ is Cauchy. Let $\epsilon > 0$. Then, there exists $N \in \mathbb{N}$ such that, for all $m > n > N$,

$$\sum_{p=1}^{m} ||x_p|| - \sum_{p=1}^{q} ||x_q|| = ||x_{n+1}|| + ||x_{n+2}|| + \ldots + ||x_m|| < \epsilon$$

Therefore, if we let $S_l = \sum_{t=1}^{l} x_t$, the sequence of partial sums of $\sum x_j$,

$||S_m - S_n|| = ||x_{n+1} + x_{n+2} + \ldots + x_m|| \leq ||x_{n+1}|| + ||x_{n+2}|| + \ldots + ||x_m|| < \epsilon.$

Hence, the sequence of partial sums of $\sum x_j$ is Cauchy. Since V is complete, the partial sums converge, meaning $\sum x_j$ converges. □

b) Show by an example that if V is not complete, a series may converge absolutely yet not converge.

Let S be the space of real sequences $x = (x_1, x_2, \ldots)$ with only a finite number of non-zero terms, and let $||x|| := \max_j |x_j|$. Then, as demonstrated in 5 below, this is an incomplete normed linear space. Let

$$x_k = (0, 0, \ldots, 0, \frac{1}{k^2}, 0, 0, \ldots).$$

Then the series $\sum x_k$ converges absolutely, since

$$\sum ||x_k|| = \sum \max_j |x_{k_j}| = \sum \frac{1}{k^2}$$

converges. However, it is clear that

$$\sum x_k = (1, \frac{1}{4}, \frac{1}{9}, \ldots) \notin S$$

since it has an infinite number of non-zero terms.

c) Conversely, if V has the property that every absolutely convergent series also converges to some element of V, then show that V is complete.

Proof. Let $\{x_j\}$ be a Cauchy sequence in V. Then, for all integers $k \geq 0$, there exists $N_k \in \mathbb{N}$ such that, for $p, q > N_k$,

$$||x_p - x_q|| < 2^{-k}.$$

Choose one such $p > N_k$ and denote x_p by x_{j_k}. Then $\{x_{j_n}\}$ is a subsequence of $\{x_j\}$ where

$$||x_{j_{n+1}} - x_{j_n}|| < 2^{-n}.$$

Then

$$\sum ||x_{j_{n+1}} - x_{j_n}||$$

converges since $\sum 2^{-n}$ does. By hypothesis,

$$\sum (x_{j_{n+1}} - x_{j_n})$$

converges, which is to say the sequence of partial sums

$$S_m = \sum_{i=1}^{m} (x_{j_{i+1}} - x_{j_i}) = x_{j_{m+1}} - x_{j_0}$$

converges. Since $\{x_{j_n} - x_{j_0}\}$ converges, it is certainly true that the subsequence $\{x_{j_n}\}$ converges. By problem 2 above, then, we can conclude that the sequence $\{x_j\}$ converges. Since our choice of a Cauchy sequence was arbitrary, we conclude that V is complete. □

4

Let X be any metric space and $\mathcal{B}(X, \mathbb{R})$ the metric space of all *bounded* real valued functions $f : X \to \mathbb{R}$ with the metric

$$\rho(f, g) := \sup_{x \in X} |f(x) - g(x)|.$$

Show that this metric space is complete.

Proof. By contradiction. Suppose $\mathcal{B}(X, \mathbb{R})$ is not complete. That is to say, there exists a Cauchy sequence $\{f_n\}$ in $\mathcal{B}(X, \mathbb{R})$ such that $\{f_n\}$ does not converge to an element of $\mathcal{B}(X, \mathbb{R})$. $\{f_n\}$ does, however, converge to an element of the completion of $\mathcal{B}(X, \mathbb{R})$, so this limit f must be unbounded.

Let M_n denote a bound for each f_n. Let $1 > \epsilon > 0$. Then, there exists $N \in \mathbb{N}$ such that, if $j, k \geq N$,

$$\epsilon > ||f_j - f_k|| = \sup_{x \in X} |f_j(x) - f_k(x)|.$$

Then, clearly, for $m \geq N$, $|f_m(x)| \leq M_N + 1$ for all $x \in X$. Now, let

$$M = \max\{M_1, M_2, \ldots, M_{N-1}, M_N + 1\}.$$

M is a bound on all of the f_n. However, since f is unbounded, there exists $x \in X$ such that $|f(x)| > M + 1$. Therefore,

$$||f_n - f|| = \sup_{x \in X} |f_n(x) - f(x)| \geq 1 > \epsilon$$

for all $n \in \mathbb{N}$. As such, the sequence $\{f_n\}$ does not converge to f. This contradiction implies that, in fact, $\mathcal{B}(X, \mathbb{R})$ is complete. □

5

Consider the linear space S of real sequences $x = (x_1, x_2, \ldots)$ with only a finite number of non-zero terms. Let $||x|| := \max_j |x_j|$.

a) Show that this is a norm on this space.

Proof. To see that this is a norm on the space, let $x \in S$. Then

$$||x|| = \max_j |x_j| \geq 0$$

because $|x_j| \geq 0$ for all j. Also, $0 = ||x_j|| = \max_j |x_j|$ implies that $|x_j| = 0$ for all j, which is to say that $x = 0$.

Also, if $c \in \mathbb{R}$,

$$||cx|| = \max_j |cx_j| = \max_j c|x_j| = c \max_j |x_j| = c||x||.$$

Finally, if $x, y \in S$,

$$||x+y|| = \max_j |x_j+y_j| \leq \max_j(|x_j|+|y_j|) = \max_j |x_j|+\max_j |y_j| = ||x||+||y||,$$

so this is, in fact, a norm on the space. □

b) Is this space complete with this norm? Justify your response.

No, this space is not complete with this norm. To justify this statement, let

$$x_n = (1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}, 0, 0, \ldots).$$

Let $\epsilon > 0$. Then let $N > \frac{1}{\epsilon}$. For all $m > n > N$,

$$\begin{array}{ll} ||x_m - x_n|| & = ||(0, 0, \ldots, 0, \frac{1}{n+1}, \frac{1}{n+2}, \ldots, \frac{1}{m}, 0, 0, \ldots)|| \\ & = \frac{1}{n+1} < \frac{1}{n} < \epsilon. \end{array}$$

So $\{x_n\}$ is a Cauchy sequence. However,

$$\lim_{n\to\infty} x_n = (1, \frac{1}{2}, \frac{1}{3}, \ldots) \notin S$$

since this limit has no zero terms.

6

Consider the space ℓ_1 of real sequences $x = (x_1, x_2, \ldots)$ that converge absolutely. Show that

$$||x|| := \sum |x_j|$$

defines a norm on this space and that with this norm the space is complete.

Proof. First, we argue that this is really a norm. To see that, let $x, y \in \ell_1$ and $c \in \mathbb{R}$. Now,

$$||x|| = \sum |x_j| \geq 0$$

since each term $|x_j| \geq 0$. Also, $0 = ||x|| = \sum |x_j|$ only if $|x_j| = 0$ for all j, meaning $x = 0$.

Also,

$$||cx|| = ||c(x_1, x_2, \ldots)|| = ||(cx_1, cx_2, \ldots)|| = \sum |cx_j| = |c| \sum |x_j| = |c|||x||.$$

Finally,

$$||x + y|| = \sum |x_j + y_j| \leq \sum (|x_j| + |y_j|) = \sum |x_j| + \sum |y_j| = ||x|| + ||y||.$$

Now, let $\sum x_k$ be an absolutely convergent series in ℓ_1. Then there exists $L \in \mathbb{R}$ such that

$$L = \sum_k ||x_k|| = \sum_k \sum_j |x_{k_j}| = \sum_j \sum_k |x_{k_j}|, \qquad (*)$$

where switching the order of summation is allowable since the series is absolutely convergent. This, in turn, implies that, for all j, $\sum_k x_{k_j}$ converges to some limit x_k. Hence,

$$\sum x_k = (\sum x_{k_1}, \sum x_{k_2}, \ldots) = (x_1, x_2, \ldots) = x.$$

Now, from (*) above, we know, in turn, that the series $\sum x_j$ converges absolutely, so $x \in \ell_1$. Thus, we can conclude that ℓ_1 is a complete space under this norm. $\square$

7

Show that a closed subset of a complete metric space is complete.

Proof. Let S be a complete metric space and let $C \subseteq S$ be a closed subset of S. Let $\{x_k\}$ be a Cauchy sequence in C. Then, since S is complete, there exists $x \in S$ such that $x_k \to x$. Now, since C is closed and, therefore, contains all of its limit points, we know that, in fact, $x \in C$. Since our choice $\{x_k\}$ was arbitrary, we see that every Cauchy sequence in C has a limit in C, meaning C is complete. □

8

Let $a = x_0 < x_1 < \ldots < x_n = b$ be a partition of the real interval $a \leq x \leq b$. A *step function* is a function that is constant on each of the subintervals.

a) If the interval $a \leq x \leq b$ is bounded, show that the step functions are dense in the space of continuous functions using the uniform norm.

Proof. Let f be continuous on $[a, b]$ and, for each $n \in \mathbb{N}$, define $x_{n_k} = x_0 + \frac{k}{n}$ and, for $x_{n_j} \leq x < x_{n_{j+1}}$, $f_n(x) = f(x_{n_j})$. Then $\{f_n\}$ is a sequence of step functions. Now, since $[a, b]$ is compact, we know that f is uniformly continuous on $[a, b]$. This means that, for $\epsilon > 0$ there exists $\delta > 0$ such that, for $x, y \in [a, b]$, if $|x - y| < \delta$, then $|f(x) - f(y)| < \epsilon$. Define $N = \frac{1}{\delta}$. Then, for $n > N$,

$$x_{n_{k+1}} - x_{n_k} < \delta$$

for all k. Hence, for $x \in [a, b]$, if $[x_{n_j}, x_{n_{j+1}}]$ is the sub-interval containing x

$$|f_n(x) - f(x)| = |f(x_{n_j}) - f(x)| < \epsilon$$

since

$$|x_{n_j} - x| < \delta.$$

Therefore,

$$||f_n - f|| = \max_{x \in [a,b]} |f_n(x) - f(x)| < \epsilon$$

so $f_n \to f$. Our choice of the continuous function f was arbitrary, so we can conclude that the step functions are dense in the space of continuous functions under the uniform norm. □

b) Is this still true if the interval is unbounded?

No, this is not necessarily true if the interval is unbounded. For example, consider the function $f(x) = x^2$ on the interval $[0, +\infty)$. Suppose there exists a sequence of step functions f_n that converge to f.

Let

$$0 = x_{n_0} < x_{n_1} < \ldots$$

be the partition of $[0, +\infty)$ associated with f_n. Let $\epsilon > 0$. Then there exists $N \in \mathbb{N}$ such that, for $k > N$,

$$\frac{\epsilon}{2} > ||f_k - f|| = \max_{x \in [0,+\infty)} |f_k(x) - f(x)|.$$

Let $l_r = x_{k_{r+1}} - x_{k_r}$. Then, since f_k is constant on each subinterval,

$$\begin{aligned} l_r x_{k_r} + \tfrac{l_r^2}{2} &= |l_r x_{k_r} + \tfrac{l_r^2}{2}| \\ &= |f(x_{k_r} + \tfrac{l_r}{4}) - f(x_{k_r} + \tfrac{3l_r}{4})| \\ &= |(f(x_{k_r} + \tfrac{l_r}{4}) - f_k(x_{k_r} + \tfrac{l_r}{4})) + (f(x_{k_r} + \tfrac{3l_r}{4}) - f_k(x_{k_r} + \tfrac{3l_r}{4}))| \\ &\leq |f(x_{k_r} + \tfrac{l_r}{4}) - f_k(x_{k_r} + \tfrac{l_r}{4})| + |f(x_{k_r} + \tfrac{3l_r}{4}) - f_k(x_{k_r} + \tfrac{3l_r}{4})| \\ &< \tfrac{\epsilon}{2} + \tfrac{\epsilon}{2} \\ &= \epsilon. \end{aligned}$$

Hence,

$$x_{k_r} < \frac{\epsilon - \frac{l_r^2}{2}}{l_r}$$

meaning the lengths of the subintervals must get arbitrarily small as x_{k_r} increases. Therefore, there is no sequence of step functions that converge to f, meaning the step functions are not necessarily dense in the continuous functions on an unbounded interval.

c) Is this still true for a bounded open interval $a < x < b$?

No, this is not necessarily true on an open interval. For example, consider the function $f(x) = \frac{1}{x}$ on the open interval $(0, 1)$. Suppose there exists a sequence of step functions f_n that converge to f. Then, if the subintervals of the partition associated with each f_n are not to get arbitrarily small, that is to say that f is uniformly continuous on $(0, 1)$. Since it is not, we conclude that the step functions are not necessarily dense in the continuous functions on an open interval.

9

If $a \leq x \leq b$ is a bounded interval, show that the piecewise linear functions are dense in the space of continuous functions using the uniform norm.

Proof. Let f be a continuous function on $[a, b]$. Then let $\{f_n\}$ be a sequence of piecewise linear functions on $[a, b]$ where the partition associated with f_k is

$$a = x_{k_0} < x_{k_1} < \ldots < x_{k_m} = b$$

such that

$$x_{k_j} = x_{k_0} + \frac{j}{m}$$

and

$$f_k(x_{k_j}) = f(x_{k_j}).$$

Let $\epsilon > 0$. Then, since $[a, b]$ is compact, f is uniformly continuous and so are each of the f_n. Hence, there exists $\delta_1 > 0$ such that, if $|x - y| < \delta_1$

$$||f(x) - f(y)|| < \frac{\epsilon}{2}$$

and, for any f_k, $\delta_2 > 0$ such that, if $|x - y| < \delta_2$,

$$||f_k(x) - f_k(y)|| < \frac{\epsilon}{2}.$$

Let $\delta = \min\{\delta_1, \delta_2\}$ and let $N = \frac{1}{\delta}$. Then, for $l > N$,

$$\begin{aligned} ||f_l - f|| &= \max_{x \in [a,b]} |f_l(x) - f(x)| \\ &= \max_{x \in [a,b]} |(f_l(x) - f_l(x_{l_s})) + (f(x_{l_s}) - f(x))| \\ &\leq \max_{x \in [a,b]} |f_l(x) - f_l(x_{l_s})| + |f(x_{l_s}) - f(x)| \\ &< \tfrac{\epsilon}{2} + \tfrac{\epsilon}{2} \\ &= \epsilon \end{aligned}$$

where $[x_{l_s}, x_{l_{s+1}}]$ is the subinterval of the partition associated with f_l containing the value of x at which $|f_l(x) - f(x)|$ is maximized. In other words, the sequence $\{f_n\}$ converges to f.

Since our choice of f was arbitrary, we conclude that the piecewise linear functions are dense in the continuous functions over $[a, b]$. $\square$

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